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Large sets of extended directed triple systems with odd orders[☆]

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ABSTRACT

For three types of triples, unordered, cyclic and transitive, the corresponding extended triple, extended triple system and their large set are introduced. The spectrum of *LEDTS*(v) for even v has been given in our paper (Liu and Kang (2009) [9]). In this paper, we shall discuss the existence problem of *LEDTS*(v) for odd v and give the almost complete conclusion: there exists an *LEDTS*(v) for any positive integer $v \neq 4$ except possible $v = 95$, 143, 167, 203, 215.

1. Introduction

Let x, y, z be distinct elements in a finite set X. A *triple* $\{x, y, z\}$ (or *cyclic triple* $\langle x, y, z\rangle$, or *transitive triple* (x, y, z)) on X is a set of three unordered pairs $\{x, y\}$, $\{y, z\}$, $\{z, x\}$ (or ordered pairs (x, y), (y, z), (z, x), or ordered pairs (x, y), (y, z), (x, z)) of X. For these (classical) triples, the elements in each pair and triple must be distinct. Breaking this restriction, we have the so-called *extended* unordered pair (or ordered pair) and *extended* triple (or extended cyclic triple, or extended transitive triple), which were firstly introduced by Johnson and Mendelsohn in 1972; see [8].

An *extended* Steiner (or Mendelsohn, or directed) triple system *ESTS*(v) (or *EMTS*(v), or *EDTS*(v)) is a pair $(X, \mathcal{A})$, where X is a v-set and $\mathcal{A}$ is a collection of extended triples (or cyclic triples, or transitive triples) on X, called *blocks*, such that every extended unordered (or ordered) pair of X belongs to exactly one block of $\mathcal{A}$. A *large set* of *ESTS*(v) (or *EMTS*(v), or *EDTS*(v)), denoted by *LESTS*(v) (or *LEMTS*(v), or *LEDTS*(v)), is a collection $\{(X, \mathcal{A}_k)\}_k$, where X is a v-set, each $(X, \mathcal{A}_k)$ is an *ESTS*(v) (or *EMTS*(v), or *EDTS*(v)) and these $\mathcal{A}_k$ form a partition of all extended triples (or cyclic triples, or transitive triples) on X. For positive integers $w < v$, let Y be a w-subset of the v-set X. An *EDTS*(v, w) is a trio $(X, Y, \mathcal{A})$, where $\mathcal{A}$ is a collection of extended transitive triples on X, called *blocks*, such that every extended ordered pair of X not belonging to Y (called *hole*) is contained exactly in one block of $\mathcal{A}$. An *LEDTS*(v, w) is a collection $\{(X, Y, \mathcal{A}_i) : 1 \leq i \leq 3v - 2\}$ such that all extended transitive triples from X, not belonging to Y, are partitioned into $\mathcal{A}_i$, $1 \leq i \leq 3v - 2$, where each $(X, Y, \mathcal{A}_i)$ is an *EDTS*(v, w) for $1 \leq i \leq 3w - 2$ or an *EDTS*(v) for $3w - 1 \leq i \leq 3v - 2$.

The types of extended triples (or cyclic triples, or transitive triples) and the extended pairs contained in them are listed in the following table.

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System	Forms of triple	Pairs covered by triple	The number of triples in a v-set	The number of systems in a large set
	$S_1 : \{x, x, x\}$	$\{x, x\}$	v	
ESTS	$S_2 : \{x, x, y\}$	$\{x, x\}, \{x, y\}$	$v(v-1)$	v
	$S_3 : \{x, y, z\}$	$\{x, y\}, \{y, z\}, \{z, x\}$	$v(v-1)(v-2)/6$	
	$M_1 : \langle x, x, x\rangle$	(x, x)	v	
EMTS	$M_2 : \langle x, x, y\rangle$	$(x, y), (y, x), (x, x)$	$v(v-1)$	v
	$M_3 : \langle x, y, z\rangle$	$(x, y), (y, z), (z, x)$	$v(v-1)(v-2)/3$	
	$D_1 : (x, x, x)$	(x, x)	v	
	$D_2 : (x, x, y)$	$(x, x), (x, y)$	$v(v-1)$	
EDTS	$D_3 : (x, y, y)$	$(x, y), (y, y)$	$v(v-1)$	$3v-2$
	$D_4 : (x, y, x)$	$(x, y), (y, x), (x, x)$	$v(v-1)$	
	$D_5 : (x, y, z)$	$(x, y), (y, z), (x, z)$	$v(v-1)(v-2)$	

The existence problem of extended Steiner triple system and extended Mendelsohn triple system have been solved in [1,2,8]. The existence problem of extended directed triple system with some additional conditions has also been discussed in [5,3]. In [9], we solved the existence problem of $LESTS(v)$ and $LEMTS(v)$. We also determine the existence of $LEDTS(v)$ for even v. In this paper, we will discuss the existence problem of *LEDTS* with odd orders and determine its spectrum.

Lemma 1.1 ([9]).

(1) *There exists an* $LESTS(v)$ *for any integer* $v \geq 1$;
(2) *There exists an* $LEMTS(v)$ *for any integer* $v \geq 1$;
(3) *For even* $v \geq 2$, *there exists an* $LEDTS(v)$ *if and only if* $v \neq 4$.

Let K be a set of positive integers, $t, v, g_1, \ldots, g_r, n_1, \ldots, n_r$ be positive integers, s be non-negative integer and $\sum_{i=1}^{r} n_i g_i = v - s$. A *candelabra t-system* (t, K)-$CS(v : s)$ or (t, K)-$CS(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$, see [11], is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ satisfying the following conditions:

(C1) X is a v-set (called *points*), S is its s-subset (called *stem*);
(C2) $\mathcal{G}$ is a partition of $X \setminus S$, which consists of n_i g_i-subsets (called *groups*), $1 \leq i \leq r$;
(C3) $\mathcal{A}$ is a family of some subsets of X, each member (called *block*) has the size from K;
(C4) Every t-subset T of X is contained in exactly one block if $|T \cap (S \cup G)| < t$, $\forall G \in \mathcal{G}$, or in no block if $T \subseteq (S \cup G)$ for some $G \in \mathcal{G}$.

A *candelabra quadruple system* $CQS(v : s)$ is a $(3, \{4\})$-$CS(v : s)$. A (t, K)-$CS(1^v : 0)$ is just a t-wise balanced design $S(t, K, v)$, briefly denoted by t-*BD*.

A *group divisible t-design* (or t-*GDD*) on a v-set is a trio $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G}$ is a 1-*BD* on X; $\mathcal{B}$ is a family of subsets of X (called *blocks*) and $|A \cap G| \leq 1$, $\forall A \in \mathcal{B}, G \in \mathcal{G}$; each t-subset from t distinct groups is contained in exactly one block. If the block sizes come from a set K, it is denoted by $GDD(t, K, v)$. The *group type* of the *GDD* means the list $\{|G| : G \in \mathcal{G}\}$. A $GDD(3, K, v) = (X, \mathcal{G}, \mathcal{B})$ is called an *s-fan* $GDD(3, (K_1, \ldots, K_s, K_{\mathcal{T}}), v)$ if $\mathcal{B}$ can be partitioned into disjoint subsets $\mathcal{B}_1, \ldots, \mathcal{B}_s$ and $\mathcal{T}$, such that each $(X, \mathcal{G}, \mathcal{B}_i)$ is a $GDD(2, K_i, v)$ for $1 \leq i \leq s$. A *frame* $F(3, 3, g^n)$ is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying that

(F1) X is a gn-set of points;
(F2) $\mathcal{G}$ is a collection of ng-subsets (called *groups*) of X which partition X;
(F3) $\mathcal{A}$ consists of all triples from X, intersecting each group in at most one points;
(F4) $\mathcal{A}$ can be partitioned into $gn\mathcal{A}_x$, $x \in G \in \mathcal{G}$, such that each $(X \setminus G, \mathcal{G} \setminus \{G\}, \mathcal{A}_x)$ is a $GDD(2, 3, g^{n-1})$.

Let v be a positive integer, $\mathcal{G}$ be a partition of a v-set X, and $K_1, \ldots, K_s, K_{\mathcal{T}}$ be the sets of positive integers. Suppose that $\mathcal{B}_1, \ldots, \mathcal{B}_s$ and $\mathcal{T}$ are collections of some subsets of X with the size from $K_1, \ldots, K_s$ and $K_{\mathcal{T}}$ respectively. An *s-fan design* s-$FG(3, (K_1, \ldots, K_s, K_{\mathcal{T}}), v)$ is an $(s+3)$-tuple $(X, \mathcal{G}, \mathcal{B}_1, \ldots, \mathcal{B}_s, \mathcal{T})$, where $(X, \mathcal{G})$ is a 1-*BD*, each $(X, \mathcal{G} \cup \mathcal{B}_i)$ is a 2-*BD* for $1 \leq i \leq s$, and $(X, \mathcal{G} \cup (\cup_{i=1}^{s} \mathcal{B}_i) \cup \mathcal{T})$ is a 3-*BD*. An s-$FG(3, (3, \ldots, 3, 4), g_1^{n_1} \cdots g_r^{n_r})$ can be obtained from a $CQS(g_1^{n_1} \cdots g_r^{n_r} : s)$ by deleting all points in its stem.

Lemma 1.2. (1) [7] *There exists a* 2-$FG(3, (\{3, 5\}, \{3, 5\}, \{4, 6\}), 2^k)$ *for any integer* $k \geq 3$.
(2) [7] *For odd* $k \in \{37, 39, 43\} \cup [57, 73] \cup [81, 157] \cup [177, \infty)$, *there exists a* 2-$FG(3, (K_4, K_4, K_4), k)$ *of type* $1^1 g_1^{\alpha_1} \cdots g_r^{\alpha_r}$, *where even* $g_i \neq 4$, $1 \leq i \leq r$, $K_4 = \{n : n \geq 4\}$.
(3) [6] *There exists a* g-$FG(3, (4, \ldots, 4, 4), g^4)$ *if* $4|g$.
(4) [13] *There exists a* 2-$FG(3, (4, 3, 3), 3^{4k+1})$ *for* $k > 0$.
(5) [14] *There exists a* 2-$FG(3, (3, 3, \{4, 6\}), 2^{k-2}4^1)$ *for* $k \equiv 2$ *mod* 3 *and* $k \geq 5$.
(6) [14] *There exists a* 2-$FG(3, (3, 3, 4), 2^k)$ *for* $k \equiv 0, 1$ *mod* 3 *and* $k \geq 3$.

(7) [4] *There exists an* $S(3, q+1, q^2+1)$ *for prime power* q.
(8) [10] *For* $n > 3$ *and* $n \neq 5$, *a* $GDD(3, 4, g^n)$ *exists if and only if* gn *is even and* $3|g(n-1)(n-2)$. *For* $n = 5$, *a* $GDD(3, 4, g^n)$ *exists if* g *is divisible by* 4 *or* 6.

Below, I_n means an n-set, Z_n means a residual ring modulo n, and F_q is a finite field of order q. Denote $Z_n^* = Z_n \setminus \{0\}$ and $F_q^* = F_q \setminus \{0\}$. Denote an extended transitive triple by (a, b, c) or abc. For a family of extended transitive triples $\mathcal{A}$ on Z_n (or F_q) and $x, m \in Z_n$ (or F_q), denote

$$\mathcal{A} + x = \{(a+x, b+x, c+x) : (a, b, c) \in \mathcal{A}\}, \qquad m\mathcal{A} = \{(ma, mb, mc) : (a, b, c) \in \mathcal{A}\},$$
$$-\mathcal{A} = \{(-a, -b, -c) : (a, b, c) \in \mathcal{A}\} \quad \text{and} \quad \mathcal{A}^{-1} = \{(c, b, a) : (a, b, c) \in \mathcal{A}\}.$$

Definition 1.1. For integers $n_i, g_i > 0,\ 1 \leq i \leq r$, a *directed group divisible triple system* $DGDD(g_1^{n_1} \cdots g_r^{n_r})$ is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying that

(1) X is a set of $\sum_{i=1}^r n_i g_i$ elements (points);
(2) $\mathcal{G}$ is a partition of X into n_i subsets (groups) of size g_i, $1 \leq i \leq r$;
(3) $\mathcal{A}$ is a family of some transitive triples (blocks) from X, intersecting each group in at most one point;
(4) Each ordered pair on X from distinct (or same) groups is contained in exactly one (or no) block.

Definition 1.2. For $n, g, s > 0$, an *extended directed group divisible triple system* $EDGDD(g^n s^1)$ is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying that

(1) X is a set of $(ng + s)$ elements (points);
(2) $\mathcal{G} = \{G_0, \ldots, G_n\}$ (groups) is a partition of X, where $|G_0| = s$, other $|G_i| = g$;
(3) $\mathcal{A}$ is a family of extended transitive triples (blocks) from X such that $A \not\subseteq G$, $\forall A \in \mathcal{A}, G \in \mathcal{G}$;
(4) Each ordered 2-subset (x, y) of X is contained in exactly one (or no) block if x, y are in distinct (or same) groups. Each pair (x, x) is contained in exactly one (or no) block if $x \notin G_0$ (or $x \in G_0$).

Definition 1.3. For $n > 0$, a $DGDD^*(3^n 1^1)$ is a trio $(X \cup \{w\}, \mathcal{G}, \mathcal{A})$ satisfying that

(1) $X \cup \{w\}$ is a set of $(3n + 1)$ elements (points), $w \notin X$;
(2) $\mathcal{G}$ is a partition of X into ordered $G_i = (a_{i,0}, a_{i,1}, a_{i,2}),\ i \in I_n$, the element of $\mathcal{G} \cup \{w\}$ is called group;
(3) $\mathcal{A}$ is a family of transitive triples (blocks) from $X \cup \{w\}$ such that $A \notin \mathcal{G}$ and each ordered pair (b, c) or $(a_{i,j}, a_{i,j+1})$, $i \in I_n, j \in Z_3$, is contained in exactly one block, where b, c belong to the distinct groups.

Definition 1.4. For $n > 0$, an $EDGDD^*(3^n 1^1)$ is a trio $(X \cup \{w\}, \mathcal{G}, \mathcal{A})$ satisfying that

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(2) $\mathcal{G}$ is a partition of X into ordered $G_i = (a_{i,0}, a_{i,1}, a_{i,2}),\ i \in I_n$, the element of $\mathcal{G} \cup \{w\}$ is called group;
(3) $\mathcal{A}$ is a family of extended transitive triples (blocks) from $X \cup \{w\}$ such that $A \not\subseteq G$, $\forall A \in \mathcal{A}, G \in \mathcal{G}$, each ordered pair (x, x) or (b, c) or $(a_{i,j+1}, a_{i,j})$, $i \in I_n, j \in Z_3$, is contained in exactly one block, where $x \in X$ and b, c belong to the distinct groups.

Example 1.1. A $DGDD^*(3^2 1^1)$, an $EDGDD(3^2 1^1)$ and an $EDGDD^*(3^2 1^1)$:

Take $X = Z_6$ and $\mathcal{G} = \{G_1, G_2, \{w\}\}$, where ordered $G_1 = (0, 1, 2)$ and $G_2 = (3, 4, 5)$.

$DGDD^*(3^2 1^1)$: 501, 412, 320, 034, 245, 153, $0w5$, $1w4$, $2w3$, $3w1$, $4w0$, $5w2$;

$EDGDD(3^2 1^1)$: 040, 131, 252, 323, 414, 505, $03w$, $15w$, $24w$, $w30$, $w51$, $w42$;

$EDGDD^*(3^2 1^1)$: 004, 113, 225, 332, 441, 550, 310, 052, 154, 423, $w21$, $w35$, $4w0$, $2w4$, $03w$, $51w$.

It is unanimous for these designs to cover all ordered pairs in distinct groups. As well, the designs $EDGDD$ and $EDGDD^*$ cover all pairs (x, x) for $x \in Z_6$. And, the design $DGDD^*$ covers the pairs (0, 1), (1, 2), (2, 0) and (3, 4), (4, 5), (5, 3), the design $EDGDD^*$ covers the pairs (0, 2), (2, 1), (1, 0) and (3, 5), (5, 4), (4, 3) too.

2. Further concepts

Definition 2.1. For integers $n, g > 0$, a $DF(g^n)$ is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying that

(1) X is a set of gn elements (points);
(2) $\mathcal{G}$ is a partition of X into n subsets (groups) of size g. For $x \in G \in \mathcal{G}$, denote $G = G_x$;
(3) $\mathcal{A}$ consists of all transitive triples (blocks) from X, intersecting each group in at most one point,
(4) $\mathcal{A}$ can be partitioned into $\{\mathcal{B}_x^j : x \in X, j \in I_3\}$, where $\mathcal{B}_x^j$ forms a $DGDD(g^{n-1})$ on $X \setminus G_x$ with the group set $\mathcal{G} \setminus \{G_x\}$.

Definition 2.2. For $n, g > 0$ and $s \geq 2$, a $PDGDD(g^n : s)$ is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ satisfying that

(1) X is a set of ng elements, S is a set of s-set and $X \cap S = \emptyset$, the element of $X \cup S$ is called point;
(2) $\mathcal{G}$ is a partition of X into n subsets (groups) of size g. For $x \in G \in \mathcal{G}$, denote $G = G_x$.
(3) $\mathcal{A}$ consists of all transitive triples (blocks) from $X \cup S$ such that $|A \cap S| \leq 1$, $|A \cap G| \leq 1$, $\forall A \in \mathcal{A}$, $G \in \mathcal{G}$;
(4) $\mathcal{A}$ can be partitioned into $\{\mathcal{A}_x^j : x \in X, j \in I_3\} \cup \{\mathcal{B}_k : 1 \leq k \leq 3(s-2)\}$, where
$\mathcal{A}_x^j$ forms a $DGDD(g^{n-1}(s+1)^1)$ on $X \setminus (G_x \setminus \{x\})$ with the group set $(\mathcal{G} \setminus \{G_x\}) \cup \{S \cup \{x\}\}$,
$\mathcal{B}_k$ forms a $DGDD(g^n)$ on $X \setminus G_0$ with the group set $\mathcal{G}$.

Example 2.1. A $PDGDD(3^4 : 2)$:

Take $X = Z_{12},\ S = \{u, v\}$ and $\mathcal{G} = \{G_i : 0 \leq i \leq 3\}$, where $G_i = \{i, i+4, i+8\}$, $0 \leq i \leq 3$. Define the following transitive triple families on $Z_{12} \setminus \{4, 8\}$ with the groups G_1, G_2, G_3 and $\{u, v, 0\}$, where 10,11 are written as $\bar{0}$, $\bar{1}$, respectively.

$\mathcal{A}_0^0$: $u96, u75, u\bar{0}\bar{1}, 9u3, \bar{1}u2, 6u1, 21u, 57u, 3\bar{0}u, 279, 392, 560, v\bar{0}5, v\bar{1}1, v76, 7v2, \bar{0}v9, 6v3, 31v, 52v,$
$9\bar{1}v, 012, 035, 067, 136, 09\bar{0}, 10\bar{1}, 25\bar{1}, \bar{0}71, 6\bar{1}5, 230, 5\bar{0}3, \bar{1}69, \bar{1}\bar{0}0, 970, 17\bar{0};$

$\mathcal{A}_0^1$: $u36, u\bar{0}9, u5\bar{1}, 3u1, 9u2, \bar{0}u7, 75u, 6\bar{1}u, 12u, 013, 027, 056, v2\bar{1}, v69, 2v3, 3v5, 7v1, 9v7, 5v\bar{0}, \bar{1}\bar{0}v,$
$16v, 617, 250, 09\bar{1}, 10\bar{0}, \bar{1}21, \bar{0}1\bar{1}, 329, 572, \bar{1}65, 3\bar{0}0, \bar{0}53, 963, 760, 79\bar{0}, \bar{1}90;$

$\mathcal{A}_0^2$: $u35, u27, u\bar{1}6, \bar{1}u1, 9u\bar{0}, \bar{0}u9, 71u, 52u, 36u, 201, 063, 709, v16, v97, v23, v5\bar{1}, 7v\bar{0}, 61v, 25v, 39v,$
$\bar{0}\bar{1}v, 130, 172, 0\bar{1}2, \bar{1}05, 96\bar{1}, 1\bar{1}\bar{0}, 932, 2\bar{1}9, \bar{0}31, 53\bar{0}, \bar{0}50, 0\bar{0}7, 576, 690, 675.$

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x,\ x \in Z_{12},\ j \in Z_3$. Then, we can prove that the collection $\{\mathcal{A}_x^j : x \in X, j \in Z_3\}$ forms a $PDGDD(3^4 : 2)$. In fact, it is not difficult to verify that each $\mathcal{A}_0^j$ is a $DGDD(3^4)$ with the group set $(\mathcal{G} \setminus \{G_0\}) \cup \{\{0, u, v\}\}$, and each $\mathcal{A}_x^j$ forms a $DGDD(3^4)$ with the group set $(\mathcal{G} \setminus \{G_i\}) \cup \{\{x, u, v\}\},\ x \in G_i \in \mathcal{G}$. And, all $\mathcal{A}_x^j$ are mutually disjoint. So, these designs form a desired $PDGDD(3^4 : 2)$ indeed. $\square$

Definition 2.3. For integers $s, g_i, n_i > 0,\ 1 \leq i \leq n$, a $PECS(g_1^{n_1} \cdots g_r^{n_r} : s)$ is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ satisfying that

(1) X is a set of $\left(\sum_{i=1}^{r} n_i g_i\right)$ elements, S is an s-set and $X \cap S = \emptyset$, the element of $X \cup S$ is called point;
(2) $\mathcal{G}$ is a partition of X into n_i subsets (groups) of size g_i. For $x \in G \in \mathcal{G}$, denote $G = G_x$;
(3) $\mathcal{A}$ consists of all extended transitive triples (blocks) from $X \cup S$, not belonging to $G \cup S,\ \forall G \in \mathcal{G}$;
(4) $\mathcal{A}$ can be partitioned into $\{\mathcal{B}_x^j : x \in X, j \in I_3\} \cup \{\mathcal{C}_k : k \in I_{3s-2}\}$, where
$\mathcal{B}_x^j$ forms an $EDGDD\left(1^{\sum_{i=1}^{r} n_i g_i - |G_x|}(|G_x| + s)^1\right)$ on $X \cup S$ with the long group $G_x \cup S$,
$\mathcal{C}_k$ forms a $DGDD(g_1^{n_1} \cdots g_r^{n_r})$ on X with the group set $\mathcal{G}$.

Definition 2.4. For integers $n, g > 0$ and $s \geq 0$, a $PECS^*(g^n : s)$ is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ satisfying that

(1) X is a set of ng elements, S is an s-set and $X \cap S = \emptyset$, the element of $X \cup S$ is called point;
(2) $\mathcal{G}$ is a partition of X into n subsets (groups) of size g. For $x \in G \in \mathcal{G}$, denote $G = G_x$;
(3) $\mathcal{A}$ consists of all extended transitive triples (blocks) from $X \cup S$, not belonging to $G \cup S,\ \forall G \in \mathcal{G}$;
(4) $\mathcal{A}$ can be partitioned into $\{\mathcal{A}_x^j : x \in X, j \in Z_3\} \cup \{\mathcal{B}_k : k \in I_{3s+4}\}$, where
$\mathcal{A}_x^j$ forms an $EDGDD(1^{g(n-1)}(g + s - 1)^1)$ on $(X \setminus \{x\}) \cup S$ with the long group $(G_x \setminus \{x\}) \cup S$,
$\mathcal{B}_k$ forms a $DGDD(g^n)$ on X with the group set $\mathcal{G}$.

Example 2.2. A $PECS^*(3^3 : 1)$:

Take $X = F_9,\ S = \{w\},\ \mathcal{G} = \{G_i : i \in Z_3\}$ and g is a primitive element of F_9, where $g^2 = 1 + 2g$ and $G_0 = \{0, g, g^5\}$, $G_1 = G_0 + 1 = \{1, g^7, g^2\}$, $G_2 = G_0 + g^3 = \{g^3, g^4, g^6\}$. Below, define the extended transitive triple families $\mathcal{A}_0^1, \mathcal{A}_0^2$ on $F_9^* \cup \{w\}$ with the long group $\{g, g^5, w\}$, and define the transitive triple families $\mathcal{B}_k$ $(k \in I_7)$ on F_9 with the group set $\mathcal{G}$. In $\mathcal{A}_0^1, \mathcal{A}_0^2, \mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$, the point g^a is briefly denoted by its index a and the point 0 is denoted by $*$.

$\mathcal{A}_0^1$: 006, 223, 332, 447, 660, 774, 461, 210, 163, 052, 645, 627, 172, 314, 071, 356, 570, 430, 753, 254, $w04, w26, w37, 76w, 03w, 42w;$

$\mathcal{A}_0^2$: 060, 232, 303, 424, 676, 747, 146, 017, 120, 257, 705, 502, 721, 613, 341, 453, 564, 365, $0w4,$ $2w6, 3w7, 4w0, 6w2, 7w3;$

$\mathcal{B}_1$: $*04$, 176, 235, $*26$, 103, 457, $*37$, 142, 560, $40*$, 532, 671, 301, $62*$, 754, 065, 241, $73*$;

$\mathcal{B}_2$: $04*$, 352, 617, 031, $26*$, 574, 056, 214, $37*$, $*40$, 253, 716, $*62$, 130, 475, $*73$, 412, 650;

$\mathcal{B}_3$: $4 * 0523$, 761, 310, $6 * 2$, 745, 421, 605, $7 * 3$, $0 * 4$, 167, 325, 013, $2 * 6$, 547, 124, $3 * 7$, 506;

$\mathcal{B}_4 = \{(1, g^6, g) + i, (g, g^6, 1) + i,\ i \in F_9\}$; $\quad \mathcal{B}_5 = \{(g, g^2, g^6) + i, (1, g, g^4) + i,\ i \in F_9\}$;

$\mathcal{B}_6 = \{(1, g^4, g) + i, (1, g^5, g^3) + i,\ i \in F_9\}$; $\quad \mathcal{B}_7 = \{(g, 1, g^6) + i, (1, g^5, g^4) + i,\ i \in F_9\}$.

Let $\mathcal{A}_0^0 = (\mathcal{A}_0^1)^{-1}$ and $\mathcal{A}_x^j = \mathcal{A}_0^j + x,\ x \in F_9,\ j \in Z_3$. Then, we can prove that the collection $\{\mathcal{A}_x^j : x \in X, j \in Z_3\} \cup \{\mathcal{B}_k : k \in I_7\}$ forms a $PECS^*(3^3 : 1)$. In fact, it is not difficult to verify that

each of $\mathcal{A}_0^1$, $\mathcal{A}_0^2$ and $\mathcal{A}_0^0$ is an $EDGDD(3^1 1^6)$ on $F_9^* \cup \{w\}$ with the long group $G_0 = \{g, g^5, w\}$,
each $\mathcal{B}_k$ is a $DGDD(3^3)$ with the group set $\mathcal{G}$.

And, all $\mathcal{A}_x^j$ and $\mathcal{B}_k$ are mutually disjoint. Therefore, these designs form a desired $PECS^*(3^3 : 1)$ indeed. □

Definition 2.5. For integer $n > 0$, a $PECS1(3^n)$ is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying that

(1) X is a set of $3n$ elements (points);
(2) $\mathcal{G}$ is a partition of X into ordered groups $G_i = (a_{i,0}, a_{i,1}, a_{i,2}),\ i \in I_n$;
(3) $\mathcal{A}$ consists of all extended transitive triples (blocks) from X with the forms : (a, b, c) or $(d, a_{i,j}, a_{i,j+1})$ or $(a_{i,j+1}, a_{i,j}, d)$ or (a, b, a), where a, b, c belong to the distinct groups and $d \notin \{a_{i,0}, a_{i,1}, a_{i,2}\}$.
(4) $\mathcal{A}$ can be partitioned into $\{\mathcal{A}_x^r : x \in X, r \in Z_3\}$, where $\mathcal{A}_x^1 = (\mathcal{A}_x^0)^{-1}$ and
$\mathcal{A}_x^0$ forms a $DGDD^*(3^{n-1}1^1)$ on $X \setminus \{a_{i,j}, a_{i,j+1}\}$ with the group set $(\mathcal{G} \setminus \{G_i\}) \cup \{a_{i,j+2}\},\ x = a_{i,j}$;
$\mathcal{A}_x^2$ forms an $EDGDD(3^{n-1}1^1)$ on $X \setminus \{a_{i,j}, a_{i,j+1}\}$ with the group set $(\mathcal{G} \setminus \{G_i\}) \cup \{a_{i,j+2}\},\ x = a_{i,j}$.

Definition 2.6. For integer $n > 0$, a $PECS2(3^n)$ is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying that

(1) X is a set of $3n$ elements (points);
(2) $\mathcal{G}$ is a partition of X into ordered groups $G_i = (a_{i,0}, a_{i,1}, a_{i,2}),\ i \in I_n$;
(3) $\mathcal{A}$ consists of all extended transitive triples (blocks) from X, with the forms:

$$(a, b, c), (a_{i,j}, d, a_{i,j+1}), (a_{i,j+1}, d, a_{i,j}), (d, a_{i,j+1}, a_{i,j}), (a_{i,j}, a_{i,j+1}, d), (a, a, b), (b, a, a),$$

where a, b, c belong to the distinct groups and $d \notin \{a_{i,0}, a_{i,1}, a_{i,2}\}$.
(4) $\mathcal{A}$ can be partitioned into $\{\mathcal{B}_x^r : x \in X, r \in Z_3\} \cup \{\mathcal{C}_k : k \in I_4\}$, where $\mathcal{B}_x^1 = (\mathcal{B}_x^0)^{-1}$ and
$\mathcal{B}_x^0$ forms an $EDGDD^*(3^{n-1}1^1)$ on $X \setminus \{a_{i,j}, a_{i,j+2}\}$ with the group set $(\mathcal{G} \setminus \{G_i\}) \cup \{a_{i,j+1}\},\ x = a_{i,j}$;
$\mathcal{B}_x^2$ forms a $DGDD(1^{3n-2})$ on $X \setminus \{a_{i,j}, a_{i,j+2}\},\ x = a_{i,j}$;
$\mathcal{C}_k$ forms a $DGDD(3^n)$ on X with the group set $\mathcal{G}$.

Example 2.3. A $PECS1(3^3)$ and a $PECS2(3^3)$:
Take $X = F_9$, $\mathcal{G} = \{G_i : i \in Z_3\}$ and g is a primitive element of F_9, where $g^2 = 1 + 2g$ and the ordered groups $G_0 = (0, g, g^5)$, $G_1 = G_0 + 1 = (1, g^7, g^2)$, $G_2 = G_0 + g^3 = (g^3, g^4, g^6)$. Below, define the following extended transitive triple families, where the point g^a is briefly denoted by its index $a \in Z_8$ and the point 0 is denoted by $*$.

$\mathcal{A}_0^0$: 034, 607, 246, 320, 472, 763, 056, 253, 357, 450, 652, 754;

$\mathcal{A}_0^2$: 040, 262, 323, 474, 606, 737, 035, 245, 530, 542, 567, 765;

$\mathcal{B}_0^0$: 004, 226, 332, 447, 660, 773, 410, 062, 423, 764, 370, 031, 214, 671, 127, 136;

$\mathcal{B}_0^2$: 012, 304, 067, 316, 147, 264, 327 and their converse;

$\mathcal{C}_1 = \{(0, g^2, g^3) + i, (g^3, g^2, 0) + i : i \in F_9\}$; $\qquad \mathcal{C}_2 = \{(0, 1, g^3) + i, (g^3, 1, 0) + i : i \in F_9\}$;
$\mathcal{C}_3 = \{(0, g^4, g^2) + i, (g^2, g^4, 0) + i : i \in F_9\}$; $\qquad \mathcal{C}_4 = \{(0, g^3, g^2) + i, (g^2, g^3, 0) + i : i \in F_9\}$.

Let $\mathcal{A}_0^1 = (\mathcal{A}_0^0)^{-1}$, $\mathcal{B}_0^1 = (\mathcal{B}_0^0)^{-1}$ and $\mathcal{A}_x^r = \mathcal{A}_0^r + x,\ \mathcal{B}_x^r = \mathcal{B}_0^r + x,\ x \in F_9,\ r \in Z_3$. It is easy to verify that

$\mathcal{A}_0^0$ is a $DGDD^*(3^2 1^1)$, $\mathcal{A}_0^2$ is an $EDGDD(3^2 1^1)$ on $F_9^* \setminus \{g\}$ with the group set $\{g^5\} \cup G_1 \cup G_2$,
$\mathcal{B}_0^0$ is an $EDGDD^*(3^2 1^1)$ on $F_9^* \setminus \{g^5\}$ with the group set $\{g\} \cup G_1 \cup G_2$,
$\mathcal{B}_0^2$ is a $DGDD(1^7)$ on $F_9^* \setminus \{g^5\}$, and $\mathcal{C}_k$ is a $DGDD(3^3)$ with the group set $\mathcal{G}$.

And, these families cover all allowed triples. Therefore, the collection $\{\mathcal{A}_x^r : x \in X, r \in Z_3\}$ forms a $PECS1(3^3)$ and the collection $\{\mathcal{B}_x^r : x \in X, r \in Z_3\} \cup \{\mathcal{C}_k : k \in I_4\}$ forms a $PECS2(3^3)$ on F_9. □

Example 2.4. A $PECS1(3^5)$ and a $PECS2(3^5)$:
Take $X = Z_{15}$ and $\mathcal{G} = \{G_i : i \in Z_5\}$, where the ordered groups $G_i = (i, i + 5, i + 10),\ i \in Z_5$. Below, the elements 10, 11, 12, 13, 14 are denoted by $\bar{0}$, $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$ respectively. Define

$\mathcal{A}_0^0$: $123, 147, 189, 1\bar{0}\bar{2}, 1\bar{3}\bar{4}, 248, 29\bar{1}, 2\bar{0}\bar{3}, 32\bar{4}, 341, 36\bar{0}, 379, 73\bar{1}, \bar{4}3\bar{2}, 4\bar{0}3, \bar{1}4\bar{3}, 6\bar{3}4, 96\bar{2}, 971, 7\bar{0}4, 7\bar{4}6,$
$8\bar{2}1, 8\bar{1}2, \bar{0}86, 8\bar{0}7, 9\bar{0}2, \bar{1}98, \bar{4}\bar{0}1, \bar{3}\bar{0}9, \bar{0}\bar{4}1, \bar{2}\bar{1}3, \bar{1}\bar{4}7, \bar{1}\bar{2}\bar{0}, \bar{2}\bar{3}6, \bar{2}\bar{4}8, \bar{4}\bar{3}2, 78\bar{3}, 46\bar{1}, 69\bar{4}, 9\bar{3}3, 8\bar{4}4, \bar{2}49, 4\bar{2}2,$
$\bar{3}7\bar{2}, 638, \bar{3}\bar{1}1, 627, 216;$

$\mathcal{A}_0^2$: $17\bar{3}, 9\bar{1}7, 26\bar{3}, 423, \bar{4}\bar{0}\bar{2}, 4\bar{0}\bar{3}, \bar{1}\bar{2}3, 46\bar{2}, 8\bar{0}\bar{1}, 289, 37\bar{0}, 784, \bar{4}18, \bar{4}\bar{1}\bar{3}, \bar{0}69, 2\bar{0}1, \bar{2}19, 6\bar{4}3, \bar{3}71, 7\bar{1}9, \bar{3}62, 324,$
$\bar{2}\bar{0}\bar{4}, \bar{3}\bar{0}4, 3\bar{2}\bar{1}, \bar{2}64, \bar{1}\bar{0}8, 982, \bar{0}73, 487, 81\bar{4}, \bar{3}\bar{1}\bar{4}, 96\bar{0}, 1\bar{0}2, 91\bar{2}, 3\bar{4}6, 131, 2\bar{1}2, 393, 414, 676, 7\bar{4}7, 868, 9\bar{3}9,$
$\bar{1}4\bar{1}, \bar{2}8\bar{2}, \bar{3}\bar{2}\bar{3}, \bar{4}2\bar{4};$

$\mathcal{B}_0^0$: $117, 221, 33\bar{1}, 442, 669, 773, 88\bar{2}, 996, \overline{11}3, \overline{22}4, \overline{33}4, \overline{44}8, \overline{23}1, 9\bar{1}7, 14\bar{3}, 26\bar{3}, 28\bar{1}, 189, \overline{12}4, 239, 79\bar{3}$, $47\bar{1}, 12\bar{4}, 781, 13\bar{2}, 36\bar{4}, 9\bar{2}8, 46\bar{2}, 324, 648, \overline{41}2, \bar{3}92, \bar{3}74, \bar{4}76, 431, \overline{32}6, 862, \bar{1}84, \overline{43}1, 87\bar{4}, \overline{42}3, \bar{2}91, 637$, $\bar{1}93, 15\bar{1}, 25\bar{2}, 35\bar{3}, 45\bar{4}, 651, 752, 853, 954, \bar{1}56, \bar{2}57, \bar{3}58, \bar{4}59$;

$\mathcal{B}_0^2$: $123, 246, 369, 159, 5\bar{1}2, 6\bar{3}5, 574, \bar{2}3\bar{1}, 9\bar{3}7, 5\bar{4}3, 4\bar{2}1, \bar{3}34, 49\bar{1}, 8\bar{3}1, \bar{4}48, \bar{1}17, 27\bar{4}, 16\bar{4}, 9\overline{42}, 387, 8\bar{2}5, \overline{413}$, $928, 7\bar{2}6, \bar{2}2\bar{3}, 6\bar{1}8$ and their reverse;

$\mathcal{C}_1 : 014, 029, 410, 920,\ \bmod 15$; $\mathcal{C}_2 : 034, 079, 430, 970,\ \bmod 15$;

$\mathcal{C}_3 : 01\bar{2}, 06\bar{3}, \bar{2}10, \bar{3}60,\ \bmod 15$; $\mathcal{C}_4 : 0\overline{12}, 07\bar{3}, \overline{21}0, \bar{3}70,\ \bmod 15$.

Using the same notation and proof with Example 2.3, we can verify that

$\mathcal{A}_0^0$ is a $DGDD^*(3^4 1^1)$, $\mathcal{A}_0^2$ is an $EDGDD(3^4 1^1)$ on $Z_{15}^* \setminus \{5\}$ with the group set $\{10\} \cup G_1 \cup G_2$,
$\mathcal{B}_0^0$ is an $EDGDD^*(3^4 1^1)$ on $Z_{15}^* \setminus \{10\}$ with the group set $\{5\} \cup G_1 \cup G_2$,
$\mathcal{B}_0^2$ is a $DGDD(1^{13})$ on $Z_{15}^* \setminus \{10\}$, and $\mathcal{C}_k$ is a $DGDD(3^5)$ with the group set $\mathcal{G}$.

And, these families cover all allowed triples. Therefore, the collection $\{\mathcal{A}_x^r : x \in Z_{15}, r \in Z_3\}$ forms a $PECS1(3^5)$ and the collection $\{\mathcal{B}_x^r : x \in Z_{15}, r \in Z_3\} \cup \{\mathcal{C}_k : k \in I_4\}$ forms a $PECS2(3^5)$ on Z_{15}. □

Definition 2.7. For integers $n, g > 0$, a $PEDGDD(g^n : 1)$ is a trio $(X \cup \{w\}, \mathcal{G}, \mathcal{B})$ satisfying that

(1) $X \cup \{w\}$ is a set of $(ng + 1)$ elements (points), $w \notin X$;
(2) $\mathcal{G}$ is a partition of X into n subsets (groups) of size g, denote $G = G_x$ for $x \in G \in \mathcal{G}$;
(3) $\mathcal{B}$ consists of all such extended transitive triples (blocks) T from $X \cup \{w\}$, that $|T| > 1$; $|T \cap G| \leq 1$, $\forall G \in \mathcal{G}$; if $w \in T$ then $|T| = 3$ and w does not appear in the middle position.
(4) $\mathcal{B}$ can be partitioned into $\{\mathcal{A}_x^j : x \in X, j \in Z_3\}$, where
$\mathcal{A}_x^0$ forms an $EDGDD(g^{n-1}1^1)$ on $(X \setminus G_x) \cup \{x\}$ with the short group $\{x\}$,
$\mathcal{A}_x^1$ (and $\mathcal{A}_x^2$) forms an $EDGDD(g^{n-1}2^1)$ on $(X \setminus G_x) \cup \{x, w\}$ with the short group $\{x, w\}$.

Example 2.5. A $PEDGDD(3^3 : 1)$:
Take $X = Z_9$ and $\mathcal{G} = \{\{i, i+3, i+6\} : i \in Z_3\}$. For $x \in G \in \mathcal{G}$, denote $G = G_x$. First, construct the following extended transitive triple families $\mathcal{A}_0^0$ (on $Z_9 \setminus \{3, 6\}$) and $\mathcal{A}_0^1, \mathcal{A}_0^2$ (on $(Z_9 \setminus \{3, 6\}) \cup \{w\}$):

$\mathcal{A}_0^0$: 121, 242, 484, 515, 757, 878, 018, 045, 027, 720, 540, 810;

$\mathcal{A}_0^1$: 115, 224, 442, 551, 778, 887, $w12$, $w57$, $w48$, $84w$, $75w$, $21w$, 180, 450, 270, 072, 054, 081;

$\mathcal{A}_0^2$: 511, 422, 244, 155, 877, 788, $w21$, $w75$, $w84$, $48w$, $57w$, $12w$, 801, 504, 702, 207, 405, 108.

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x$, $x \in Z_9$, $j \in Z_3$. It is not difficult to verify that

$\mathcal{A}_0^0$ forms an $EDGDD(1^1 3^2)$ on $Z_9 \setminus \{3, 6\}$ with the groups $\{0\} \cup G_1 \cup G_2$;
$\mathcal{A}_0^1$ (and $\mathcal{A}_0^2$) forms an $EDGDD(2^1 3^2)$ on $(Z_9 \setminus \{3, 6\}) \cup \{w\}$ with the groups $\{0, w\} \cup G_1 \cup G_2$.

And, these families cover all allowed triples. Therefore, the collection $\{\mathcal{A}_x^j : x \in Z_9, j \in Z_3\}$ forms a $PEDGDD(3^3 : 1)$. □

Definition 2.8. For integers $n, g > 0$, a $PDCS(g^n : 1)$ is a triple $(X \cup \{w\}, \mathcal{G}, \mathcal{C})$ satisfying that

(1) $X \cup \{w\}$ is a set of $(ng + 1)$ elements (points), $w \notin X$;
(2) $\mathcal{G}$ is a partition of X into n subsets (groups) of size g, denote $G = G_x$ for $x \in G \in \mathcal{G}$;
(3) $\mathcal{C}$ consists of all transitive triples (blocks) T from $X \cup \{w\}$ such that $T \not\subseteq G \cup \{w\}$, $\forall G \in \mathcal{G}$ and if $w \in T$ then w only appears in the middle position.
(4) $\mathcal{C}$ can be partitioned into $\{\mathcal{A}_x^j : x \in X, j \in Z_3\} \cup \{\mathcal{B}\}$, where
$\mathcal{A}_x^0$ forms a $DGDD(1^{g(n-1)}g^1)$ on $(X \setminus \{x\}) \cup \{w\}$ with the long group $(G_x \setminus \{x\}) \cup \{w\}$,
$\mathcal{A}_x^1$ ($\mathcal{A}_x^2$) forms a $DGDD(1^{g(n-1)}(g-1)^1)$ on $X \setminus \{x\}$ with the long group $G_x \setminus \{x\}$,
$\mathcal{B}$ forms a $DGDD(g^n)$ on X with the group set $\mathcal{G}$.

Example 2.6. A $PDCS(3^3 : 1)$:
Take $X = F_9$ and $\mathcal{G} = \{\{x, x+g, x+g^5\} : x = 0, 1, g^3\}$, where g is a primitive element of F_9, $g^2 = 1 + 2g$. For $x \in G \in \mathcal{G}$, denote $G = G_x$. First, construct $\mathcal{A}_0^0$ on $F_9^* \cup \{w\}$, construct $\mathcal{A}_0^1$ and $\mathcal{A}_0^2$ on F_9^*, and construct $\mathcal{B}$ on F_9 as follows. In the $\mathcal{A}_0^j$ ($j \in Z_3$), the point g^a is briefly denoted by its index $a \in Z_8$.

$\mathcal{A}_0^0$: $2w3$, $7w4$, $6w7$, $4w2$, $3w0$, $0w6$, 416, 520, 721, 254, 276, 632, 356, 537, 645, 601, 102, 043, 705, 407, 173, 314;

$\mathcal{A}_0^1$: 012, 034, 056, 307, 140, 136, 217, 250, 243, 426, 631, 532, 735, 741, 457, 654, 670, 762;

$\mathcal{A}_0^2$: 013, 024, 065, 507, 710, 126, 174, 420, 621, 723, 275, 630, 341, 352, 367, 453, 476, 564;

$\mathcal{B} = \{(1, g^3, g^5) + x,\ (1, g, g^6) + x : x \in F_9\}$.

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x,\ x \in Z_9,\ j \in Z_3$. It is not difficult to verify that,

$\mathcal{A}_0^0$ forms a $DGDD(3^1 1^6)$ on $F_9^* \cup \{w\}$ with the long group $\{w, g, g^5\}$;
$\mathcal{A}_0^1$ (and $\mathcal{A}_0^2$) forms a $DGDD(2^1 1^6)$ on F_9^* with the long group $\{g, g^5\}$;
$\mathcal{B}$ forms a $DGDD(3^3)$ on F_9 with the group set $\mathcal{G}$.

And, these families cover all allowed triples. Therefore, the collection $\{\mathcal{A}_x^j : x \in Z_9, j \in Z_3\} \cup \{\mathcal{B}\}$ forms a $PDCS(3^3 : 1)$. $\square$

3. Recursive construction

Lemma 3.1 ([9]). *There exists a $DF(g^n)$ satisfying one of the following conditions for positive integers g, n:*

(1) $n \equiv 1, 2 \bmod 3$; (2) $6|n$ and $3|g$; (3) $n \equiv 3 \bmod 6$, $n > 3$ and $6|g$.

Theorem 3.1. *If there exist a $PECS(g_0^1 g_1^{n_1} \cdots g_r^{n_r} : s)$, an $LEDTS(g_i + s, s)$ for $1 \le i \le r$, and an $LEDTS(g_0 + s)$, then there exist an $LEDTS\left(\sum_{i=1}^r g_i n_i + g_0 + s\right)$, an $LEDTS\left(\sum_{i=1}^r g_i n_i + g_0 + s, g_0 + s\right)$ and an $LEDTS\left(\sum_{i=1}^r g_i n_i + g_0 + s, s\right)$.*

Proof. Let $PECS(g_0^1 g_1^{n_1} \cdots g_r^{n_r} : s) = (X, S, \mathcal{G}, \mathcal{A})$, where $|X| = \sum_{i=1}^r g_i n_i + g_0 + s$, $|S| = s$, $\mathcal{G}$ consists of one g_0-group G_0 and $n_i g_i$-groups, $1 \le i \le r$. $\mathcal{A}$ consists of all extended transitive triples from X, not belonging to $S \cup G, \forall G \in \mathcal{G}$. $\mathcal{A}$ can be partitioned into $\{\mathcal{B}_x^j : x \in G \in \mathcal{G}, j \in I_3\} \cup \{\mathcal{C}_k : k \in I_{3s-2}\}$, where each $\mathcal{B}_x^j$ forms an $EDGDD(1^{\sum_{i=1}^r n_i g_i + g_0 - |G|}(|G| + s)^1)$ on X with the long group $G \cup S,\ x \in G \in \mathcal{G}$, each $\mathcal{C}_k$ forms a $DGDD(g_0^1 g_1^{n_1} \cdots g_r^{n_r})$ on $X \setminus S$ with the group set $\mathcal{G}$.

By the assumption, there exists an $LEDTS(|G| + s, s)$ on $G \cup S$ for each $G \in \mathcal{G} \setminus \{G_0\}$, which contains

$3|G|$ disjoint $EDTS(|G| + s) = (G \cup S, \mathcal{C}_x^j)$, $\quad x \in G, j \in I_3$;
$(3s - 2)$ disjoint $EDTS(|G| + s, s) = (G \cup S, S, \mathcal{D}_k(G))$, $\quad 1 \le k \le 3s - 2$.

And, there exists an $LEDTS(g_0 + s)$ on $G_0 \cup S$. It contains

$3|G_0|$ disjoint $EDTS(g_0 + s) = (G_0 \cup S, \mathcal{C}_x^j)$, $\quad x \in G_0, j \in I_3$;
$(3s - 2)$ disjoint $EDTS(g_0 + s) = (G_0 \cup S, \mathcal{E}_k)$, $\quad 1 \le k \le 3s - 2$.

Define $\Gamma_x^j = \mathcal{B}_x^j \cup \mathcal{C}_x^j,\ x \in G \in \mathcal{G}, j \in I_3$ and $\Lambda_k = (\bigcup_{G \in \mathcal{G} \setminus \{G_0\}} \mathcal{D}_k(G)) \cup \mathcal{C}_k \cup \mathcal{E}_k,\ 1 \le k \le 3s - 2$. Then each Γ_x^j or Λ_k forms an $EDTS\left(\sum_{i=1}^r g_i n_i + g_0 + s\right)$ on X, and they form an $LEDTS\left(\sum_{i=1}^r g_i n_i + g_0 + s\right)$. Obviously, there also exist an $LEDTS\left(\sum_{i=1}^r g_i n_i + g_0 + s, g_0 + s\right)$ and an $LEDTS\left(\sum_{i=1}^r g_i n_i + g_0 + s, s\right)$ by the structure of the obtained $LEDTS\left(\sum_{i=1}^r g_i n_i + g_0 + s\right)$. $\square$

Theorem 3.2. *If there exist 2-$FG(3, (K_{\mathcal{B}}, K_{\mathcal{C}}, K_{\mathcal{D}}), g^n)$, $PECS1(3^k)\ \forall\, k \in K_{\mathcal{B}}$, $PECS2(3^k)\ \forall\, k \in K_{\mathcal{C}}$, and $DF(3^k)\ \forall\, k \in K_{\mathcal{D}}$, then there exists a $PECS^*((3g)^n : 0)$.*

Construction. Let 2-$FG(3, (K_{\mathcal{B}}, K_{\mathcal{C}}, K_{\mathcal{D}}), g^n) = (X, \mathcal{G}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, where $\mathcal{G}$ is a partition of gn-set X into ng-groups. Denote $S' = S \times Z_3$ for $S \subseteq X$ and $\mathcal{G}_A = \{G_x : x \in A\}$ for $A \subseteq X$, where $G_x = \{x\} \times Z_3$. The point $(x, i) \in X \times Z_3$ is briefly written as x_i. By assumption, we have the following designs (1)–(3):

(1) $PECS1(3^{|A|}) = (A', \mathcal{G}_A, \mathcal{B}_A)$ for $A \in \mathcal{B}$. The $\mathcal{B}_A$ can be partitioned into $9|A|$ disjoint $\mathcal{B}_{x,j}^r(A), x \in A, j, r \in Z_3$, where $\mathcal{B}_{x,j}^0(A)$ ($\mathcal{B}_{x,j}^2(A)$) is a $DGDD^*(3^{|A|-1}1^1)$ ($EDGDD(3^{|A|-1}1^1)$) on $A' \setminus \{x_j, x_{j+1}\}$ with the groups $(\mathcal{G}_A \setminus \{G_x\}) \cup \{x_{j+2}\}$, and $\mathcal{B}_{x,j}^1(A) = (\mathcal{B}_{x,j}^0(A))^{-1}$.

(2) $PECS2(3^{|A|}) = (A', \mathcal{G}_A, \mathcal{C}_A)$ for $A \in \mathcal{C}$. The $\mathcal{C}_A$ can be partitioned into $9|A|$ disjoint $\mathcal{C}_{x,j}^r(A)$ and four disjoint $\mathcal{C}_k(A)$, $x \in A, j, r \in Z_3, k \in I_4$, where $\mathcal{C}_{x,j}^1(A) = (\mathcal{C}_{x,j}^0(A))^{-1}$ and

$\mathcal{C}_{x,j}^0(A)$ is an $EDGDD^*(3^{|A|-1}1^1)$ on $A' \setminus \{x_j, x_{j+2}\}$ with the groups $(\mathcal{G}_A \setminus \{G_x\}) \cup \{x_{j+1}\}$;
$\mathcal{C}_{x,j}^2(A)$ is a $DGDD(1^{3|A|-2})$ on $A' \setminus \{x_j, x_{j+2}\}$;
$\mathcal{C}_k(A)$ is a $DGDD(3^{|A|})$ on A' with the groups $\mathcal{G}_A$.

(3) $DF(3^{|A|}) = (A', \mathcal{G}_A, \mathcal{D}_A)$ for $A \in \mathcal{D}$. The $\mathcal{D}_A$ can be partitioned into $9|A|$ disjoint $\mathcal{D}_{x,j}^r(A)$, which is a $DGDD(3^{|A|-1})$ on $(A \setminus \{x\}) \times Z_3$ with the groups $\mathcal{G}_{A \setminus \{x\}},\ x \in A, j, r \in Z_3$.

Now, define

$$\mathcal{F}_{x,j}^r = \left(\bigcup_{x \in A \in \mathcal{B}} \mathcal{B}_{x,j}^r(A)\right) \cup \left(\bigcup_{x \in A \in \mathcal{C}} \mathcal{C}_{x,j}^r(A)\right) \cup \left(\bigcup_{x \in A \in \mathcal{D}} \mathcal{D}_{x,j}^r(A)\right), \quad x \in X, j, r \in Z_3;$$

$$\mathcal{F}_k = \bigcup_{A \in \mathcal{C}} \mathcal{C}_k(A), \quad k \in I_4.$$

Then, $\mathcal{F} = \{\mathcal{F}_{x,j}^r : x \in X, j, r \in Z_3\} \cup \{\mathcal{F}_k : k \in I_4\}$ forms a $PECS^*((3g)^n : 0)$ on X' with the groups $\{G' : G \in \mathcal{G}\}$.

Proof. (1) Each $\mathcal{F}_{x,j}^0$ (or $\mathcal{F}_{x,j}^1$) forms an $EDGDD(1^{3g(n-1)}(3g-1)^1)$ on $(X \times Z_3) \setminus \{x_j\}$ with the long group $G' \setminus \{x_j\}$, $x \in G \in \mathcal{G}$. Let us show that any pair $P \not\subset G' \setminus \{x_j\}$ appears exactly in one block of $\mathcal{F}_{x,j}^0$.

$\underline{P = (x_{j+2}, \beta_r) \text{ (or } (\beta_r, x_{j+2})), r \in Z_3, \beta \neq x}$. Consider the block A in $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$, $\{x, \beta\} \subset A$:

* There exists a unique block $A \in \mathcal{B}$, and $\mathcal{B}_{x,j}^0(A)$ is a $DGDD^*(3^{|A|-1}1^1)$ on $A' \setminus \{x_j, x_{j+1}\}$ with the groups $(\mathcal{G}_A \setminus \{G_x\}) \cup \{x_{j+2}\}$, so the pair P appears in a unique block $\in \mathcal{B}_{x,j}^0(A)$;
* There exists a unique block $A \in \mathcal{C}$, but the corresponding $\mathcal{C}_{x,j}^0(A)$ is defined in $A' \setminus \{x_j, x_{j+2}\}$, so the pair $P = (x_{j+2}, \beta_r)$ cannot appear in $\mathcal{C}_{x,j}^0(A)$;
* There exists a block $A \in \mathcal{D}$, but the corresponding $\mathcal{D}_{x,j}^0(A)$ is defined in $(A \setminus \{x\}) \times Z_3$, so the pair $P = (x_{j+2}, \beta_r)$ cannot appear in $\mathcal{D}_{x,j}^0(A)$.

$\underline{P = (x_{j+1}, \beta_r) \text{ (or } (\beta_r, x_{j+1})), r \in Z_3, \beta \neq x}$. Similar to the above case, we can show to exist a unique block $A \in \mathcal{C}$ such that $\{x, \beta\} \subset A$, and $\mathcal{C}_{x,j}^0(A)$ is an $EDGDD^*(3^{|A|-1}1^1)$ on $X \setminus \{x_j, x_{j+2}\}$ with the groups $(\mathcal{G}_{\mathcal{A}} \setminus \{G_x\}) \cup \{x_{j+1}\}$, so the pair P appears in a unique block $\in \mathcal{C}_{x,j}^0(A)$.

$\underline{P = (\beta_r, \beta_{r+1}), r \in Z_3, \beta \neq x}$. Consider the block A in $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$, $\{x, \beta\} \subset A$:

* There exists a unique block $A \in \mathcal{B}$, and $\mathcal{B}_{x,j}^0(A)$ is a $DGDD^*(3^{|A|-1}1^1)$ on $A' \setminus \{x_j, x_{j+1}\}$ with the groups $(\mathcal{G}_A \setminus \{G_x\}) \cup \{x_{j+2}\}$, so the pair P appears in a unique block $\in \mathcal{B}_{x,j}^0(A)$ by the definition of $DGDD^*(3^{|A|-1}1^1)$;
* There exists a unique block $A \in \mathcal{C}$, but the corresponding $\mathcal{C}_{x,j}^0(A)$ is an $EDGDD^*(3^{|A|-1}1^1)$ on $A' \setminus \{x_j, x_{j+2}\}$, which contains no pair (α_j, α_{j+1}) for any $\alpha \in A \setminus \{x\}$ by the definition of $EDGDD^*(3^{|A|-1}1^1)$;
* There exists a block $A \in \mathcal{D}$, but the corresponding $\mathcal{D}_{x,j}^0(A)$ is a $DGDD(3^{|A|-1})$, so the pair $P = (\beta_r, \beta_{r+1})$ cannot appear in $\mathcal{D}_{x,j}^0(A)$.

$\underline{P = (\beta_{r+1}, \beta_r) \text{ or } (\beta_r, \beta_r), r \in Z_3, \beta \neq x}$. Similar to the case $P = (\beta_r, \beta_{r+1})$, we can show to exist a unique block $A \in \mathcal{C}$ such that $\{x, \beta\} \subset A$, and $\mathcal{C}_{x,j}^0(A)$ is an $EDGDD^*(3^{|A|-1}1^1)$ on $X \setminus \{x_j, x_{j+2}\}$ with the groups $(\mathcal{G}_{\mathcal{A}} \setminus \{G_x\}) \cup \{x_{j+1}\}$, so the pair $P = (\beta_{r+1}, \beta_r)$ appears in a unique block $\in \mathcal{C}_{x,j}^0(A)$.

$\underline{P = (\alpha_r, \beta_s), r, s \in Z_3, \alpha \neq \beta \text{ and } x \notin \{\alpha, \beta\}}$. There exists a unique block $A \supseteq \{x, \alpha, \beta\}$ and

$A \in \mathcal{B}$ or $\mathcal{C}$ or $\mathcal{D} \Longrightarrow$ there exists a unique block in $\mathcal{B}_{x,j}^0(A)$ or $\mathcal{C}_{x,j}^0(A)$ or $\mathcal{D}_{x,j}^0(A)$ containing P, where the latter is due to $((A \setminus \{x\}) \times Z_3, \mathcal{G}_{A\setminus\{x\}}, \mathcal{D}_{x,j}^0(A)) = DGDD(3^{|A|-1})$.

(2) Each $\mathcal{F}_{x,j}^2$ forms an $EDGDD(1^{3g(n-1)}(3g-1)^1)$ on $(X \times Z_3) \setminus \{x_j\}$ with the long group $G' \setminus \{x_j\}$, $x \in G \in \mathcal{G}$. Imitating the proof method in (1), we can show that any pair $P \not\subset G' \setminus \{x_j\}$ appears exactly in one block of $\mathcal{F}_{x,j}^2$ as follows. Note that $\mathcal{B}_{x,j}^2(A)$ is an $EDGDD(3^{|A|-1}1^1)$ on $A' \setminus \{x_j, x_{j+1}\}$ with the groups $(\mathcal{G}_A \setminus \{G_x\}) \cup \{x_{j+2}\}$; $\mathcal{C}_{x,j}^2(A)$ is a $DGDD(1^{3|A|-2})$ on $A' \setminus \{x_j, x_{j+2}\}$; $\mathcal{D}_{x,j}^2(A)$ is a $DGDD(3^{|A|-1})$ on $(A \setminus \{x\}) \times Z_3$ with the groups $\mathcal{G}_{A\setminus\{x\}}$, $x \in A, j \in Z_3, r \in I_3$.

$\underline{P = (x_{j+2}, \beta_r) \text{ (or } (\beta_r, x_{j+2}), \text{ or } (\beta_r, \beta_r)), r \in Z_3, \beta \neq x}$. P appears in a unique block of $\mathcal{B}_{x,j}^2(A)$;

$\underline{P = (x_{j+1}, \beta_r) \text{ (or } (\beta_r, x_{j+1})), r \in Z_3, \beta \neq x}$. P appears in a unique block of $\mathcal{C}_{x,j}^2(A)$;

$\underline{P = (\beta_r, \beta_s), r \neq s \in Z_3, \beta \neq x}$. P appears in a unique block of $\mathcal{C}_{x,j}^2(A)$;

$\underline{P = (\alpha_r, \beta_s), r, s \in Z_3, \alpha \neq \beta \text{ and } x \notin \{\alpha, \beta\}}$. P appears in a unique block of $\mathcal{B}_{x,j}^2(A)$ (or $\mathcal{C}_{x,j}^2(A)$, or $\mathcal{D}_{x,j}^2(A)$), if $\{x, \alpha, \beta\}$ appears in a block of $\mathcal{B}$ (or $\mathcal{C}$, or $\mathcal{D}$).

(3) Each $\mathcal{F}_k$ forms a $DGDD((3g)^n)$ on X'. In fact, for any pair $P = \{\alpha_r, \beta_s\}$ from distinct groups, there exists a unique block $A \in \mathcal{C}$ containing α, β. And, by the construction, $\mathcal{C}_k(A)$ forms a $DGDD(3^{|A|})$ on A' with the groups $\mathcal{G}_A$. So, there exists a unique block in $\mathcal{C}_k(A) \subset \mathcal{F}_k$, containing P.

(4) Any triple $T = (\alpha_r, \beta_s, \gamma_t)$ belongs to $\mathcal{F}$, where $\alpha, \beta, \gamma \in X$, $\{\alpha, \beta, \gamma\} \not\subset G \in \mathcal{G}$, $r, s, t \in Z_3$.

$\underline{|\{\alpha, \beta, \gamma\}| = 3}$, i.e., α, β, γ belong to exact three distinct groups of $\mathcal{G}$. By the definition of 2-$FG(3, (K_{\mathcal{B}}, K_{\mathcal{C}}, K_{\mathcal{D}}), g^n)$, there exists a unique $A \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ such that $\{\alpha, \beta, \gamma\} \subseteq A$. Therefore, $T \in \mathcal{B}_A$ (or $\mathcal{C}_A$, or $\mathcal{D}_A$)$\subset \mathcal{F}$, if $A \in \mathcal{B}$ (or $\mathcal{C}$, or $\mathcal{D}$), by the definition of $\mathcal{B}_A$ (or $\mathcal{C}_A$, or $\mathcal{D}_A$).

$\underline{|\{\alpha, \beta, \gamma\}| = 2}$, i.e., α, β, γ belong to exact two distinct groups of $\mathcal{G}$.

* $T = (\beta_s, \alpha_r, \alpha_{r+1})$ (or $(\alpha_{r+1}, \alpha_r, \beta_s)$). There exists a unique block $A \in \mathcal{B}$ containing $\{\alpha, \beta\}$. Since $\mathcal{B}_A$ forms a $PECS1(3^{|A|})$ on A', there exists a unique $T \in \mathcal{B}_{x,j}^0(A)$ (or $\mathcal{B}_{x,j}^1(A)$) $\subset \mathcal{B}_A \subset \mathcal{F}$, where $x \neq \alpha, j \in Z_3$.
* $T = (\beta_s, \alpha_{r+1}, \alpha_r), (\alpha_{r+1}, \beta_s, \alpha_r), (\alpha_r, \alpha_{r+1}, \beta_s), (\alpha_r, \beta_s, \alpha_{r+1})$. There exists a unique block $A \in \mathcal{C}$ containing $\{\alpha, \beta\}$. Since $\mathcal{C}_A$ forms a $PECS2(3^{|A|})$ on A', there exists a unique $T \in \mathcal{C}_{x,j}^t(A) \subset \mathcal{C}_A \subset \mathcal{F}$, where $j, t \in Z_3, x \neq \alpha$.
* $T = (\alpha_r, \beta_s, \alpha_r)$ (or $(\alpha_r, \alpha_r, \beta_s), (\beta_s, \alpha_r, \alpha_r)$). There exists a unique block $A \in \mathcal{B}$ (or $\mathcal{C}$) containing $\{\alpha, \beta\}$. Since $\mathcal{B}_A$ (or $\mathcal{C}_A$) forms a $PECS1(3^{|A|})$ (or $PECS2(3^{|A|})$) on A', there exists a unique $T \in \mathcal{B}_{x,j}^2(A) \subset \mathcal{B}_A$ (or $\mathcal{C}_{x,j}^r(A) \subset \mathcal{C}_A$) $\subset \mathcal{F}$, where $x \neq \alpha, j \in Z_3, r = 0, 1$. $\square$

Theorem 3.3. *If there exist a 2-FG$(3, (K_{\mathcal{B}}, K_{\mathcal{C}}, K_{\mathcal{D}}), g_1^{n_1} \cdots g_r^{n_r})$, a PDCS$(m^k : 1)$ $\forall\, k \in K_{\mathcal{B}}$, a PEDGDD$(m^k : 1)$ $\forall\, k \in K_{\mathcal{C}}$, and a DF$(m^k)$ $\forall\, k \in K_{\mathcal{D}}$, then there exists a PECS$((mg_1)^{n_1} \cdots (mg_r)^{n_r} : 1)$.*

Construction. Let 2-$FG(3, (K_{\mathcal{B}}, K_{\mathcal{C}}, K_{\mathcal{D}}), g_1^{n_1} \cdots g_r^{n_r}) = (X, \mathcal{G}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, where $\mathcal{G}$ is a partition of $\sum_{i=1}^r g_i n_i$-set X into $n_i g_i$-groups ($i \in I_r$). Denote $\mathcal{G}_A = \{\{x\} \times I_m : x \in A\}$ for $A \subseteq X$. Take $w \notin X \times I_m$. By assumption, we have the following designs (1)–(3):

(1) $PDCS(m^{|A|} : 1) = ((A \times I_m) \cup \{w\}, \mathcal{G}_A, \mathcal{B}_A)$ for $A \in \mathcal{B}$. The $\mathcal{B}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{B}_{x,i}^j(A)$ and one $\mathcal{B}(A)$, $x \in A$, $i \in I_m$, $j \in I_3$, where

$\mathcal{B}_{x,i}^1(A)$ forms a $DGDD(1^{m(|A|-1)} m^1)$ on $((A \times I_m) \cup \{w\}) \setminus \{x_i\}$ with the long group $(\{x\} \times (I_m \setminus \{i\})) \cup \{w\}$,
$\mathcal{B}_{x,i}^2(A)$ ($\mathcal{B}_{x,i}^3(A)$) forms a $DGDD(1^{m(|A|-1)}(m-1)^1)$ on $(A \times I_m) \setminus \{x_i\}$ with the long group $\{x\} \times (I_m \setminus \{i\})$,
$\mathcal{B}(A)$ forms a $DGDD(m^{|A|})$ on $A \times I_m$ with the groups $\mathcal{G}_A$.

(2) $PEDGDD(m^{|A|} : 1) = ((A \times I_m) \cup \{w\}, \mathcal{G}_A, \mathcal{C}_A)$ for $A \in \mathcal{C}$. The $\mathcal{C}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{C}_{x,i}^j(A)$, $x \in A$, $i \in I_m$, $j \in I_3$, where

$\mathcal{C}_{x,i}^1(A)$ forms an $EDGDD(m^{|A|-1}1^1)$ on $((A \setminus \{x\}) \times I_m) \cup \{x_i\}$ with the groups $\{x_i\} \cup \mathcal{G}_{A\setminus\{x\}}$,
$\mathcal{C}_{x,i}^2(A)$ ($\mathcal{C}_{x,i}^3(A)$) forms an $EDGDD(m^{|A|-1}2^1)$ on $((A \setminus \{x\}) \times I_m) \cup \{\{w, x_i\}\}$ with the groups $\{w, x_i\} \cup \mathcal{G}_{A\setminus\{x\}}$.

(3) $DF(m^{|A|}) = (A \times I_m, \mathcal{G}_A, \mathcal{D}_A)$ for $A \in \mathcal{D}$. The $\mathcal{D}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{D}_{x,i}^j(A)$, $x \in A$, $i \in I_m$, $j \in I_3$, where each $\mathcal{D}_{x,i}^j(A)$ is a $DGDD(m^{|A|-1})$ on $(A \setminus \{x\}) \times I_m$ with the groups $\mathcal{G}_{A\setminus\{x\}}$. Define

$$\mathcal{F}_{x,i}^j = \left(\bigcup_{x\in A\in\mathcal{B}} \mathcal{B}_{x,i}^j(A)\right) \cup \left(\bigcup_{x\in A\in\mathcal{C}} \mathcal{C}_{x,i}^j(A)\right) \cup \left(\bigcup_{x\in A\in\mathcal{D}} \mathcal{D}_{x,i}^j(A)\right), \quad x \in X,\ i \in I_m,\ j \in I_3; \qquad \mathcal{F} = \bigcup_{A\in\mathcal{B}} \mathcal{B}(A).$$

Then, the collection $\{\mathcal{F}_{x,i}^j : x \in X, i \in I_m, j \in I_3\} \cup \{\mathcal{F}\}$ forms a $PECS((mg_1)^{n_1} \cdots (mg_r)^{n_r} : 1)$ on $(X \times I_m) \cup \{w\}$ with the groups $\{G \times I_m : G \in \mathcal{G}\}$ and the stem $\{w\}$.

Proof. The proof is similar to Theorem 3.2. $\square$

Theorem 3.4. *If there exists an e-$FG(3, (4, \ldots, 4, 4), g^n)$, then there exists a $PECS(g^n : e-1)$.*

Proof. Let e-$FG(3, (4, \ldots, 4), g^n) = (X, \mathcal{G}, \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{e-1}, \mathcal{T})$, where $\mathcal{G}$ is a partition of the gn-set X into ng-groups. Let $S = \{\infty_1, \ldots, \infty_{e-1}\}$. In the following (1) and (2), the symbol $A \to \{0, 1, 2, 3\}$ means a replacement: a, b, c, d in block A by 0, 1, 2, 3 respectively. Then, we have to make the anti-replacement for the constructed blocks.

(1) For each block $A = \{a, b, c, d\} \in \mathcal{A}_0$, let $A \to \{0, 1, 2, 3\}$, construct the following extended transitive triple families $\{\mathcal{B}_x^j(A) : x \in A, j \in I_3\}$ on A, where $\mathcal{B}_x^2(A) = (\mathcal{B}_x^1(A))^{-1}$, and

$$\mathcal{B}_0^1(A) = \{(0,2,1), (1,0,3), (3,2,0), (1,1,2), (2,2,3), (3,3,1)\}, \qquad \mathcal{B}_2^1(A) = \{(a,b,c) + 2 : (a,b,c) \in \mathcal{B}_0^1(A)\},$$
$$\mathcal{B}_1^1(A) = \{(1,3,2), (2,1,0), (0,3,1), (2,3,3), (3,0,0), (0,2,2)\}, \qquad \mathcal{B}_3^1(A) = \{(a,b,c) + 2 : (a,b,c) \in \mathcal{B}_2^1(A)\},$$
$$\mathcal{B}_0^3(A) = \{(1,2,1), (2,3,2), (3,1,3)\}, \qquad \mathcal{B}_1^3(A) = \{(0,3,0), (2,0,2), (3,2,3)\},$$
$$\mathcal{B}_2^3(A) = \{(0,1,0), (1,3,1), (3,0,3)\}, \qquad \mathcal{B}_3^3(A) = \{(0,2,0), (1,0,1), (2,1,2)\}.$$

(2) For each block $A = \{a, b, c, d\} \in \mathcal{A}_1$, let $A \to \{0, 1, 2, 3\}$, construct the following transitive triple families $\{\mathcal{D}_x^j(A) : x \in A, j \in I_3\} \cup \{\mathcal{C}(\mathcal{A})\}$ on $A \cup \{\infty_1\}$, where $\mathcal{D}_x^j(A) = \mathcal{D}_0^j(A) + x$, $x \in Z_4$, and

$$\mathcal{D}_0^1(A) = \{(2,1,\infty_1), (3,\infty_1,1), (\infty_1,3,2), (1,2,3)\}, \qquad \mathcal{D}_0^2(A) = \{(1,2,\infty_1), (3,\infty_1,2), (\infty_1,3,1), (2,1,3)\},$$
$$\mathcal{D}_0^3(A) = \{(1,3,\infty_1), (2,\infty_1,3), (\infty_1,1,2), (3,1,0), (2,0,1), (0,3,2)\}, \qquad \mathcal{C}(A) = \{(0,1,3) \bmod 4\}.$$

(3) For $A \in \mathcal{A}_i$, $2 \le i \le e-1$, there exists a $DF(1^5)$ on $A' = A \cup \{\infty_i\}$ by Lemma 3.1(1). Its blocks can be partitioned into 15 disjoint $\mathcal{E}_x^j(i, A)$, each is a $DGDD(1^4)$ on $A' \setminus \{x\}$, $x \in A'$, $j \in I_3$.

(4) For $A \in \mathcal{T}$, there exists a $DF(1^4)$ on A by Lemma 3.1(1). Its blocks can be partitioned into 12 disjoint $\mathcal{H}_x^j(A)$, each is a $DGDD(1^3)$ on $A \setminus \{x\}$, $x \in A$, $j \in I_3$.

Now, define

$$\mathcal{F}_x^j = \left(\bigcup_{x\in A\in\mathcal{A}_0} \mathcal{B}_x^j(A)\right) \cup \left(\bigcup_{x\in A\in\mathcal{A}_1} \mathcal{D}_x^j(A)\right) \cup \left(\bigcup_{2\le i\le e-1} \bigcup_{x\in A\in\mathcal{A}_i} \mathcal{E}_x^j(A, i)\right) \cup \left(\bigcup_{x\in A\in\mathcal{T}} \mathcal{H}_x^j(A)\right) \quad \text{for } x \in X, j \in I_3,$$
$$\mathcal{F}_0 = \bigcup_{A\in\mathcal{A}_1} \mathcal{C}(A), \qquad \mathcal{F}_{3(i-2)+j} = \mathcal{E}_{\infty_i}^j(i, A) \quad \text{for } 2 \le i \le e-1.$$

We can check that each $\mathcal{F}_x^j$ forms an $EDGDD(1^{g(n-1)}(g+e-1)^1)$ on $X \cup S$ with the long group $G \cup S$ and each $\mathcal{F}_k$ forms a $DGDD(g^n)$ on X with the groups $\mathcal{G}$, $x \in G \in \mathcal{G}$, $k \in Z_{3e-5}$. Then, the collection $\{\mathcal{F}_x^j : x \in X, j \in I_3\} \cup \{\mathcal{F}_k : 0 \le k \le 3e-6\}$ forms a $PECS(g^n : e-1)$ on $X \cup S$ with the groups $\mathcal{G}$ and stem S. $\square$

Theorem 3.5. *If there exist* 2-$FG(3, (K_{\mathcal{B}}, K_{\mathcal{C}}, K_{\mathcal{D}}), g_1^{n_1} \cdots g_t^{n_t})$, $PECS^*(m^k : r)\ \forall\, k \in K_{\mathcal{B}}$, $PDGDD(m^k : s)\ \forall\, k \in K_{\mathcal{C}}$ *and* $DF(m^k)\ \forall\, k \in K_{\mathcal{D}}$, *then there exists a* $PECS((mg_1)^{n_1} \cdots (mg_t)^{n_t} : r+s)$.

Proof. The theorem is similar to Theorem 2.3 in [9]. They differ only in whether the size of the groups in 2-*FG* are consistent. The proof is similar. □

Theorem 3.6 ([12]). *If there exists an LEDTS(v) then there exist an LEDTS$(3v)$ and an LEDTS$(3v, 3)$ for $v \geq 3,\ v \neq 6$.*

4. *LEDTS*(v) of small orders

Lemma 4.1. *There exists an LEDTS(k) for $k = 1, 3, 5, 7$.*

Proof. (1) $LEDTS(1) = \{(Z_1, \mathcal{A})\}$, where $\mathcal{A} = \{(0, 0, 0)\}$.

(2) $LEDTS(3) = \{(I_3, \mathcal{E}_j) : 0 \leq j \leq 6\}$, where

$$\mathcal{E}_0 = \{111, 222, 333, 123, 321\}, \qquad \mathcal{E}_1 = \{121, 232, 313\}, \qquad \mathcal{E}_2 = \{131, 212, 323\},$$
$$\mathcal{E}_3 = \{113, 332, 122, 231\}, \qquad \mathcal{E}_5 = \{331, 112, 322, 213\}, \qquad \mathcal{E}_4 = \mathcal{E}_3^{-1}, \qquad \mathcal{E}_6 = \mathcal{E}_5^{-1}.$$

(3) $LEDTS(5) = \{(Z_5, \mathcal{A}_x) : x \in Z_5\} \cup \{(Z_5, \mathcal{B}_k) : k \in Z_4\} \cup \{(Z_5, \mathcal{C}_k) : k \in Z_4\}$, where

$$\mathcal{A}_0 = \{000, 131, 212, 343, 424, 104, 203, 302, 401\}, \qquad \mathcal{A}_x = \mathcal{A}_0 + x, \quad x \in Z_5;$$
$$\mathcal{B}_0 = \{001, 032 \bmod 5\}, \qquad \mathcal{B}_1 = \{002, 014 \bmod 5\}, \qquad \mathcal{B}_2 = \{003, 041 \bmod 5\}, \qquad \mathcal{B}_3 = \{004, 023 \bmod 5\};$$
$$\mathcal{C}_0 = \{011, 320 \bmod 5\}, \qquad \mathcal{C}_1 = \{022, 140 \bmod 5\}, \qquad \mathcal{C}_2 = \{033, 410 \bmod 5\}, \qquad \mathcal{C}_3 = \{044, 230 \bmod 5\}.$$

(4) $LEDTS(7) = \{(F_7, \mathcal{A}_0 + x) : x \in F_q\} \cup \{(F_7, g^k\mathcal{B}_0) : k \in Z_6\} \cup \{(F_7, g^k\mathcal{C}_0) : k \in Z_6\}$, where $g = 3$ is a primitive element of the finite field F_7. And, the element g^a is denoted by its index a, the element 0 is denoted by $*$ in $\mathcal{A}_0$, $\mathcal{B}_0$ and $\mathcal{C}_0$.

$$\mathcal{A}_0 : {*}{*}{*}, 001, 112, 223, 334, 445, 550, 413, 524, 035, 140, 251, 302, 0*4, 1*5, 2*0, 3*1, 4*2, 5*3;$$
$$\mathcal{B}_0 : 1{*}{*}, 500, 411, 232, 313, {*}44, 055, 201, 340, 152, 254, 104, 453, 42{*}, 35{*}, {*}03, {*}51, 0*2;$$
$$\mathcal{C}_0 : {*}0{*}, 400, 101, 242, 303, 044, 5*5, 415, 543, 250, 052, 132, 34{*}, 351, {*}23, 2*1, 1*4. \quad \square$$

Lemma 4.2. *There exist an LEDTS$(11, 2)$, an LEDTS$(11, 3)$ and an LEDTS(11).*

Proof. Let $X = Z_8 \cup \{a, b, c\}$. First, construct an $LEDTS(11, 3) = \{(X, \mathcal{A}_x^j) : x \in Z_8, j \in Z_3\} \cup \{(X, \{a, b, c\}, \mathcal{B}_k) : k \in Z_7\}$, where all $\mathcal{A}_0^j$ and $\mathcal{B}_k$ are listed as follows, $\mathcal{A}_x^j = \mathcal{A}_0^j + x,\ x \in Z_8, j \in Z_3$, and two underlined blocks in $\mathcal{A}_0^1$ are changed into $(0, 4, 4)$ and $(4, 0, 0)$ when $\mathcal{A}_0^1 \to \mathcal{A}_x^1$ for $4 \leq x \leq 7$.

$$\begin{array}{lccccccccccc}
\mathcal{A}_0^0: & 700 & a11 & 522 & 343 & 424 & b55 & 636 & c77 & aa2 & bb0 & cc1 \\
 & 035 & 164 & 157 & 731 & 401 & 621 & 756 & 465 & 302 & 253 & 047 \\
 & a67 & a50 & b13 & b27 & c20 & c54 & 37b & 12b & 60b & 51c & 72c \\
 & 06c & 26a & 74a & 10a & ab4 & 5ba & ac3 & 3ca & bc6 & 4cb & \\
\mathcal{A}_0^1: & \underline{004} & 11a & 722 & 533 & \underline{440} & 655 & 66b & 77c & 2aa & 3bb & 1cc \\
 & 025 & 063 & 143 & 751 & 562 & 326 & 231 & 760 & 574 & 647 & 415 \\
 & 127 & a35 & a16 & a07 & b37 & b20 & b61 & c73 & c01 & c46 & 10b \\
 & 42b & 24c & 30c & 34a & 50a & 7ab & ba4 & ca2 & 6ac & cb5 & 5bc \\
\mathcal{A}_0^2: & 000 & 1a1 & 252 & 323 & 404 & 5b5 & 6c6 & 717 & a4a & b4b & c7c \\
 & 012 & 135 & 472 & 274 & 206 & 654 & 570 & 346 & 516 & 431 & 037 \\
 & 763 & a62 & a53 & b67 & b10 & c14 & c05 & 61b & 30b & 45c & 21c \\
 & 60a & 75a & a7b & b2a & a0c & c3a & b3c & c2b & & &
\end{array}$$

$$\begin{array}{llllllllllllll}
\mathcal{B}_0: & 0a5 & 0b6 & 0c7 & 022 & 034 & \bmod 8; & \mathcal{B}_1: & 0a3 & 0b2 & 0c4 & 001 & 065 & \bmod 8; \\
\mathcal{B}_2: & 0a2 & 0b5 & 0c6 & 007 & 014 & \bmod 8; & \mathcal{B}_3: & 0a6 & 0b3 & 0c1 & 002 & 054 & \bmod 8; \\
\mathcal{B}_4: & 0a7 & 0b4 & 0c2 & 003 & 056 & \bmod 8; & \mathcal{B}_5: & 0a4 & 0b1 & 0c5 & 006 & 072 & \bmod 8; \\
\mathcal{B}_6: & 0a1 & 0b7 & 0c3 & 005 & 026 & \bmod 8. & & & & & & &
\end{array}$$

It is easy to check that each $\mathcal{A}_x^j$ $(x \in Z_8, j \in Z_3)$ is an $EDTS(11)$ on X and each $\mathcal{B}_k$ $(k \in Z_7)$ is an $EDTS(11, 3)$ on X with a hole $\{a, b, c\}$. So, they form an $LEDTS(11, 3)$.

Let $\overline{\mathcal{B}}_k = \mathcal{B}_k \cup \mathcal{E}_k,\ k \in Z_7$, where $\{(\{a, b, c\}, \mathcal{E}_k) : k \in Z_7\}$ is the $LEDTS(3)$ in Lemma 4.1. Then, each $\overline{\mathcal{B}}_k$ is an $EDTS(11)$, thus $\{(X, \mathcal{A}_x^j) : x \in Z_8, j \in Z_3\} \cup \{(X, \overline{\mathcal{B}}_k) : k \in Z_7\}$ form an $LEDTS(11)$.

Furthermore, define $\mathcal{C}_x^0 = \mathcal{A}_x^0$ for $x \in Z_8$,

$$\mathcal{C}_x^i = \mathcal{A}_x^i \quad \text{for } x \in Z_8^* \setminus \{4\} \text{ and } i = 1, 2;$$
$$\mathcal{C}_0^1 = \mathcal{A}_0^1 \setminus \{(0, 0, 4), (4, 4, 0)\}, \qquad \mathcal{C}_4^1 = \mathcal{A}_4^1 \setminus \{(0, 4, 4), (4, 0, 0)\};$$
$$\mathcal{C}_0^2 = \mathcal{A}_0^2 \setminus \{(0, 0, 0), (4, 0, 4)\}, \qquad \mathcal{C}_4^2 = \mathcal{A}_4^2 \setminus \{(4, 4, 4), (0, 4, 0)\}.$$

Obviously, each of $\mathcal{C}_0^1, \mathcal{C}_4^1, \mathcal{C}_0^2, \mathcal{C}_4^2$ is the blocks of an $EDTS(11, 2)$ on X with the hole $\{0, 4\}$. Therefore, $\{(X, \mathcal{C}_x^j) : (x, j) \in (Z_8 \times Z_3) \setminus M\} \cup \{(X, \overline{\mathcal{B}}_k) : k \in Z_7\} \cup \{(X, \{0, 4\}, \mathcal{C}_x^j) : (x, j) \in M\}$ forms an $LEDTS(11, 2)$, where $M = \{(0, 1), (0, 2), (4, 1), (4, 2)\}$. □

Lemma 4.3. *There exists an LEDTS(13).*

Construction. Let $X = Z_{11} \cup \{u, v\}$. Define the following $EDTS(13)$s, where $10 \in Z_{11}$ is denoted by a.

$\mathcal{A}_0^0$:	$u\,9\,0$	$u\,3\,a$	$u\,5\,2$	$u\,8\,7$	$4\,u\,4$	$6\,7\,u$	$2\,5\,u$	$a\,3\,u$	$0\,9\,u$	$u\,1\,u$	$u\,6\,v$		
	$v\,2\,3$	$v\,4\,6$	$v\,5\,a$	$v\,1\,7$	$0\,v\,0$	$3\,8\,v$	$4\,a\,v$	$7\,5\,v$	$2\,1\,v$	$v\,8\,u$	$v\,9\,v$		
	$a\,a\,a$	$1\,a\,1$	$6\,9\,6$	$2\,2\,7$	$3\,3\,2$	$5\,5\,9$	$7\,7\,8$	$8\,8\,3$	$9\,9\,5$	$0\,1\,2$	$1\,3\,5$	$0\,3\,6$	$0\,4\,8$
	$4\,2\,0$	$6\,5\,4$	$9\,8\,a$	$0\,a\,7$	$2\,a\,9$	$4\,3\,7$	$8\,2\,4$	$a\,6\,8$	$7\,9\,3$	$8\,0\,5$	$2\,8\,6$	$6\,3\,1$	$5\,7\,6$
	$7\,1\,4$	$9\,7\,2$	$5\,3\,0$	$8\,9\,1$	$a\,4\,5$	$7\,a\,0$	$6\,a\,2$	$1\,6\,0$	$3\,9\,4$	$5\,1\,8$	$4\,1\,9$		
$\mathcal{A}_0^1$:	$u\,6\,6$	$u\,a\,2$	$u\,1\,5$	$a\,u\,0$	$1\,u\,3$	$4\,u\,7$	$6\,u\,4$	$9\,u\,8$	$0\,7\,u$	$5\,2\,u$	$3\,u\,u$	$8\,u\,v$	
	$v\,5\,5$	$v\,0\,3$	$v\,a\,7$	$5\,v\,6$	$0\,v\,2$	$7\,v\,4$	$a\,v\,8$	$2\,v\,1$	$1\,4\,v$	$6\,3\,v$	$v\,u\,9$	$9\,v\,v$	
	$3\,3\,6$	$9\,9\,0$	$a\,9\,a$	$2\,4\,2$	$1\,0\,0$	$4\,1\,1$	$5\,7\,7$	$7\,8\,8$	$8\,4\,4$	$0\,4\,5$	$0\,9\,1$	$8\,6\,2$	$1\,2\,7$
	$7\,6\,0$	$8\,7\,3$	$0\,6\,8$	$2\,8\,0$	$3\,9\,2$	$1\,8\,9$	$1\,6\,a$	$a\,6\,1$	$8\,5\,a$	$3\,4\,8$	$3\,7\,1$	$3\,5\,0$	$5\,9\,4$
	$2\,6\,5$	$7\,a\,5$	$7\,2\,9$	$4\,9\,6$	$6\,9\,7$	$9\,5\,3$	$5\,8\,1$	$2\,3\,a$	$a\,4\,3$	$4\,0\,a$			
$\mathcal{A}_0^2$:	$u\,2\,3$	$u\,5\,a$	$u\,9\,4$	$u\,8\,6$	$4\,u\,1$	$2\,2\,u$	$9\,0\,u$	$1\,6\,u$	$a\,5\,u$	$8\,7\,u$	$u\,u\,0$	$u\,v\,7$	
	$v\,1\,5$	$v\,a\,6$	$v\,2\,0$	$v\,4\,3$	$6\,v\,9$	$7\,7\,v$	$9\,a\,v$	$0\,2\,v$	$4\,8\,v$	$5\,1\,v$	$3\,v\,u$	$v\,v\,8$	
	$4\,4\,2$	$9\,9\,6$	$1\,9\,1$	$5\,6\,5$	$3\,6\,6$	$4\,a\,a$	$7\,0\,0$	$9\,3\,3$	$a\,8\,8$	$3\,9\,8$	$5\,7\,3$	$2\,a\,1$	$0\,3\,5$
	$3\,1\,2$	$6\,2\,7$	$0\,8\,1$	$7\,5\,9$	$2\,8\,5$	$2\,6\,4$	$a\,7\,2$	$6\,3\,a$	$1\,a\,4$	$7\,6\,1$	$6\,8\,0$	$4\,5\,0$	$5\,8\,4$
	$0\,4\,6$	$1\,7\,8$	$1\,3\,0$	$8\,2\,9$	$3\,7\,4$	$9\,5\,2$	$4\,9\,7$	$0\,7\,a$	$8\,a\,3$	$a\,0\,9$			
$\mathcal{B}_0$:	$u\,u\,u$	$v\,u\,v$	$0\,5\,0$	$0\,u\,4$	$0\,v\,7$	$0\,1\,3$	$3\,1\,0$	(mod 11);					
$\mathcal{B}_1$:	$u\,v\,u$	$v\,v\,v$	$0\,6\,0$	$0\,u\,7$	$0\,v\,4$	$0\,2\,3$	$3\,2\,0$	(mod 11);					
$\mathcal{B}_2$:	$u\,u\,v$	$v\,v\,u$	$0\,4\,0$	$0\,u\,5$	$0\,v\,6$	$0\,1\,9$	$0\,3\,2$	(mod 11);					
$\mathcal{B}_3$:	$v\,u\,u$	$u\,v\,v$	$0\,7\,0$	$0\,u\,6$	$0\,v\,5$	$0\,8\,9$	$0\,a\,2$	(mod 11).					

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x$, $x \in Z_{11}, j \in Z_3$, then it is not difficult to verify that each $\mathcal{B}_k$ or $\mathcal{A}_0^j$ (then $\mathcal{A}_x^j$) is an $EDTS(13)$ on X. Therefore, the collection $\{(X, \mathcal{A}_x^j) : x \in Z_{11}, j \in Z_3\} \cup \{(X, \mathcal{B}_k) : k \in Z_4\}$ forms an $LEDTS(13)$. □

5. Existence of *LEDTS*$(6t + 1)$ and *LEDTS*$(6t + 3)$

Theorem 5.1. *There exists an* $LEDTS(6k + 1)$ *for any integer* $k \geq 0$.

Proof. For $k = 0, 1, 2$, there exists an $LEDTS(6k + 1)$ by Lemmas 4.1 and 4.3.

When $k \geq 3$, there exist 2-$FG(3, (3, 3, 4), 2^k)$ for $k \equiv 0, 1 \bmod 3$ and 2-$FG(3, (3, 3, \{4, 6\}), 2^{k-2}4^1)$ for $k \equiv 2 \bmod 3$ by Lemma 1.2(6) and (5). There exist $PEDGDD(3^3 : 1)$, $PDCS(3^3 : 1)$, $DF(3^4)$ and $DF(3^6)$ by Examples 2.5 and 2.6 and Lemma 3.1(1), (2). So, we can get a $PECS(6^k : 1)$ for $k \equiv 0, 1 \bmod 3$ and a $PECS(6^{k-2}12^1 : 1)$ for $k \equiv 2 \bmod 3$ by Theorem 3.3. Since there exists an $LEDTS(7)$ which is also an $LEDTS(7, 1)$ and an $LEDTS(13)$ by Lemmas 4.1 and 4.3, there exists an $LEDTS(6k + 1)$ for any integer $k \geq 3$ by Theorem 3.1. □

Theorem 5.2. *There exists an* $LEDTS(6k + 3)$ *for any integer* $k \geq 0$.

Proof. For $k = 0, 1, 2$, there exist an $LEDTS(3)$ and an $LEDTS(5)$ by Lemma 4.1, so there exist an $LEDTS(9)$ and an $LEDTS(15)$ by Theorem 3.6.

When $k \geq 3$, there exist 2-$FG(3, (3, 3, 4), 2^k)$ for $k \equiv 0, 1 \bmod 3$ and 2-$FG(3, (3, 3, \{4, 6\}), 2^{k-2}4^1)$ for $k \equiv 2 \bmod 3$ by Lemma 1.2(6) and (5). There exist a $PECS^*(3^3 : 1)$ by Example 2.2, a $PDGDD(3^3 : 2)$ by [9], a $DF(3^4)$ and a $DF(3^6)$ by Lemma 3.1(1), (2). So, we can get a $PECS(6^k : 3)$ for any integer $k \equiv 0, 1 \bmod 3$ and a $PECS(6^{k-2}12^1 : 3)$ for any integer $k \equiv 2 \bmod 3$ by Theorem 3.5. Since there exist an $LEDTS(3)$ and an $LEDTS(5)$ by Lemma 4.1, there exist an $LEDTS(9)$, an $LEDTS(9, 3)$ and an $LEDTS(15)$ by Theorem 3.6. Then we can get an $LEDTS(6k + 3)$ for $k \geq 3$ by Theorem 3.1. □

6. Existence of *LEDTS*$(6t + 5)$*s*

Lemma 6.1. *There exist* $PECS^*(6^k : 0)$ *and* $PDGDD(6^k : 5)$ *for integer* $k \geq 3$.

Proof. There exists a 2-$FG(3, (\{3, 5\}, \{3, 5\}, \{4, 6\}), 2^k)$ for $k \geq 3$, by Lemma 1.2(1). For $i = 3, 5$, there exist a $PECS1(3^i)$ and a $PECS2(3^i)$ by Examples 2.3 and 2.4, and there exists a $DF(3^{i+1})$ by Lemma 3.1(1) and (2). Then, there exists a $PECS^*(6^k : 0)$ by Theorem 3.2.

For given $k \geq 3$, by Lemma 1.2(8), there exists a $GDD(3, 4, 6^{k+1})$. Deleting its one group, we can get a 6-*fan* $GDD(3, (3, 3, 3, 3, 3, 4), 6^k) = \left(X, \mathcal{G}', \mathcal{T} \cup \left(\bigcup_{1 \leq i \leq 6} \mathcal{A}_i\right)\right)$. Construct a $PDGDD(6^k : 5)$ on $X \cup S$ with the groups $\mathcal{G} = \mathcal{G}' \cup S$, where $S = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$, $\mathcal{G}' = \{G_i : 1 \leq i \leq k\}$. In the following (1)–(3), the symbol $A \to \{0, 1, 2\}$ means a replacement: a, b, c in block A by 0, 1, 2 respectively. Then, we have to make the anti-replacement for the constructed blocks.

(1) For each block $A = \{a, b, c\} \in \mathcal{A}_1$, let $A \to \{0, 1, 2\}$, construct the following transitive triple families $\{\mathcal{B}_x^j(A) : x \in A, j \in I_3\}$ on $A \cup \{\infty_1\}$, where $\mathcal{B}_x^j(A) = \mathcal{B}_0^j(A) + x, x \in Z_3$, and

$$\mathcal{B}_0^1(A) = \{(\infty_1, 1, 2), (2, 1, \infty_1)\}, \qquad \mathcal{B}_0^2(A) = \{(\infty_1, 2, 1), (1, 2, \infty_1)\}, \qquad \mathcal{B}_0^3(A) = \{(0, 1, 2), (2, 1, 0)\}.$$

(2) For each block $A = \{a, b, c\} \in \mathcal{A}_2$, let $A \to \{0, 1, 2\}$, construct the following transitive triple families $\{\mathcal{C}_x^j(A) : x \in A, j \in I_3\}$ on $A \cup \{\infty_2\}$, where $\mathcal{C}_x^j(A) = \mathcal{C}_0^j(A) + x, x \in Z_3$, and

$$\mathcal{C}_0^1(A) = \{(\infty_2, 1, 2), (2, 1, \infty_2)\}, \qquad \mathcal{C}_0^2(A) = \{(0, 1, 2), (2, 1, 0)\}, \qquad \mathcal{C}_0^3(A) = \{(\infty_2, 2, 1), (1, 2, \infty_2)\}.$$

(3) For each block $A = \{a, b, c\} \in \mathcal{A}_3$, let $A \to \{0, 1, 2\}$, construct the following transitive triple families $\{\mathcal{D}_x^j(A) : x \in A, j \in I_3\}$ on $A \cup \{\infty_1, \infty_2\}$, where $\mathcal{D}_x^j(A) = \mathcal{D}_0^j(A) + x, x \in Z_3$, and

$$\mathcal{D}_0^1(A) = \{(0, 1, 2), (2, 1, 0)\}, \qquad \mathcal{D}_0^2(A) = \{(1, \infty_2, 2), (2, \infty_2, 1)\}, \qquad \mathcal{D}_0^3(A) = \{(1, \infty_1, 2), (2, \infty_1, 1)\}.$$

(4) For each block $A \in \mathcal{A}_i$ $(4 \le i \le 6)$, by Lemma 3.1, there exists a $DF(1^4)$ on $A \cup \{\infty_{i-1}\}$. Its blocks can be partitioned into 12 disjoint $\mathcal{E}_x^j(i, A)$, each one is a $DGDD(1^3)$ on $(A \cup \{\infty_{i-1}\}) \setminus \{x\},\ x \in A \cup \{\infty_{i-1}\}, j \in I_3$.

(5) For each block $A \in \mathcal{T}$, by Lemma 3.1, there exists a $DF(1^4)$ on A. Its blocks can be partitioned into 12 disjoint $\mathcal{H}_x^j(A)$, each one is a $DGDD(1^3)$ on $A \setminus \{x\},\ x \in A, j \in I_3$.

Now, for $x \in X$ and $j \in I_3$, define

$$\mathcal{F}_x^j = \left(\bigcup_{x \in A \in \mathcal{A}_1} \mathcal{B}_x^j(A)\right) \cup \left(\bigcup_{x \in A \in \mathcal{A}_2} \mathcal{C}_x^j(A)\right) \cup \left(\bigcup_{x \in A \in \mathcal{A}_3} \mathcal{D}_x^j(A)\right) \cup \left(\bigcup_{4 \le i \le 6} \bigcup_{x \in A \in \mathcal{A}_i} \mathcal{E}_x^j(i, A)\right) \cup \left(\bigcup_{x \in A \in \mathcal{T}} \mathcal{H}_x^j(A)\right),$$

$$\mathcal{F}_{\infty_{i-1}}^j = \bigcup_{A \in \mathcal{A}_i} \mathcal{E}_{\infty_{i-1}}^j(i, A) \quad \text{for } 4 \le i \le 6.$$

We can check that each $\mathcal{F}_x^j$ forms a $DGDD(6^k)$ on $X \setminus (G_i \setminus \{x\})$ with the groups $(\mathcal{G} \setminus \{S, G_i\}) \cup \{S \cup \{x\}\}, x \in G_i$, and each $\mathcal{F}_{\infty_{i-1}}^j$ forms a $DGDD(6^k)$ on $X \setminus S$ with the groups $\mathcal{G}'$. Therefore, the collection $\{\mathcal{F}_x^j : x \in X, j \in I_3\} \cup \{\mathcal{F}_{\infty_{i-1}}^j : 4 \le i \le 6, j \in I_3\}$ forms the desired $PDGDD(6^k : 5)$ on $X \cup S$ with the groups $\mathcal{G}$. □

Theorem 6.1. *There exist an $LEDTS(12k + 5)$ and an $LEDTS(12k + 5, 5)$ for non-negative integer $k \ne 2$.*

Proof. Since there exists a 4-$FG(3, (4, 4, 4, 4, 4), 4^4)$ which is also a 2-$FG(3, (4, 4, 4), 4^4)$ by Lemma 1.2(3), there exists a $PECS(4^4 : 1)$ by Theorem 3.4. There exists an $LEDTS(5)$ which is also an $LEDTS(5, 1)$ by Lemma 4.1, so there exist an $LEDTS(17)$ and an $LEDTS(17, 5)$ by Theorem 3.1.

There exists a 2-$FG(3, (\{3, 5\}, \{3, 5\}, \{4, 6\}), 2^k)$ for $k \ge 3$ by Lemma 1.2(1). For $i = 3, 5$, there exist a $PECS^*(6^i : 0)$ and a $PDGDD(6^i : 5)$ by Lemma 6.1 and a $DF(6^{i+1})$ by Lemma 3.1(1), (2). Then, a $PECS(12^k : 5)$ for any $k \ge 3$ can be obtained by Theorem 3.5. From the existence of $LEDTS(17)$ and $LEDTS(17, 5)$, there exist $LEDTS(12k + 5)$ and $LEDTS(12k + 5, 5)$ for $k \ge 3$ by Theorem 3.1. □

Theorem 6.2. *There exists an $LEDTS(6k + 5)$ for odd $k \in \{37, 39, 43\} \cup [57, 73] \cup [81, 157] \cup [177, \infty)$.*

Proof. Let $K_4 = \{k \ge 4\}$. For odd $k \in \{37, 39, 43\} \cup [57, 73] \cup [81, 157] \cup [177, \infty)$, there exists a 2-$FG(3, (K_4, K_4, K_4), k)$ of type $1^1 g_1^{\alpha_1} \cdots g_r^{\alpha_r}$, where $g_i \ne 4$ is even (see Lemma 1.2(2)). Since there exist a $PECS^*(6^n : 0)$, $PDGDD(6^n : 5)$ and $DF(6^n)$ for any $n \in K_4$ by Lemmas 6.1, 3.1(1)–(3), there exists a $PECS(6^1 (6g_1)^{\alpha_1} \cdots (6g_r)^{\alpha_r} : 5)$ by Theorem 3.5. But, there exists an $LEDTS(6g_i + 5, 5),\ 1 \le i \le r$, by Theorem 6.1, and there exists an $LEDTS(11)$ by Lemma 4.2, so the conclusion holds by Theorem 3.1. □

Theorem 6.3. *There exist an $LEDTS(36t + 11)$ and an $LEDTS(36t + 11, 3)$ for $t \ge 0$. Especially, there exists an $LEDTS(6k + 5)$ for $k \in \{7, 13, 19, 25, 31, 49, 55, 79, 163, 169, 175\}$.*

Proof. By Lemma 1.2(4), there exists a 2-$FG(3, (3, 4, 4), 3^{4t+1})$ for $t > 0$. And, there exist a $PECS^*(3^3 : 0)$ by [9] and a $PDGDD(3^4 : 2)$ by Example 2.1, a $DF(3^4)$ and a $DF(3^6)$ by Lemma 3.1(1), (2), we can get a $PECS(9^{4t+1} : 2)$ by Theorem 3.5. Since there exist an $LEDTS(11, 2)$ and an $LEDTS(11)$ by Lemma 4.2, there exists an $LEDTS(36t + 11)$ for $t \ge 0$ by Theorem 3.1. Then, we can get the conclusion. By the construction of $LEDTS(11)$ in Lemma 4.2, there exists an $LEDTS(36t + 11, 3)$ for $t \ge 0$. □

Theorem 6.4. *There exists an $LEDTS(144t + 35)$ for $t \ge 0$. Especially, there exists an $LEDTS(6k + 5)$ for $k \in \{5, 29, 53, 77, 173\}$.*

Proof. By Lemma 1.2(3) there exists a $(36t + 8)$-$FG(3, (4, \ldots, 4, 4), (36t + 8)^4)$ for $t \ge 0$ which is also a 4-$FG(3, (4, 4, 4, 4, 4), (36t + 8)^4)$, so there exists a $PECS((36t + 8)^4 : 3)$ by Theorem 3.4. Further, there exist an $LEDTS(36t + 11)$ and an $LEDTS(36t + 11, 3)$ by Theorem 6.3, we can get an $LEDTS(144t + 35)$ for $t \ge 0$ by Theorem 3.1. So, there exists $LETDS(6k + 5)$ for $k \in \{5, 29, 53, 77, 173\}$. □

Theorem 6.5. *There exists an* $LEDTS(6k+5)$ *for* $k \in \{17, 21\}$.

Proof. By Lemma 1.2(3) there exists a 24-$FG(3, (4, \dots, 4, 4), 24^4)$ and a 32-$FG(3, (4, \dots, 4, 4), 32^4)$ which are also a 12-$FG(3, (4, \dots, 4, 4), 24^4)$ and a 4-$FG(3, (4, 4, 4, 4, 4), 32^4)$, so there exist a $PECS(24^4 : 11)$ and a $PECS(32^4 : 3)$ by Theorem 3.4. From Theorem 6.4, there exists an $LEDTS(35)$, and by its construction, an $LEDTS(35, 11)$ and an $LEDTS(35, 3)$ also exist. So there exist an $LEDTS(107)$ and an $LEDTS(131)$ by Theorem 3.1. □

Let q be an odd prime power, g be a primitive element of the finite field F_q, so $F_q = \{0, 1, g, g^2, \dots, g^{q-2}\}$. For a transitive triple $B = (a, b, c)$ and $x \in F_q, y \in F_q^*$, denote $B + x = (a + x, b + x, c + x)$ and $yB = (ya, yb, yc)$. All transitive triples consisted by distinct elements are partitioned into $q-2$ orbits T_i, and each T_i is separated into q sub-orbits $T_i(x)$, each T_i can also be separated into $q-1$ sub-orbits $H_i(j)$ as follows.

$$T_i = \cup\{T_i(x) : x \in F_q\}, \qquad H_i(j) = \{g^j(1 + x, x, g^i + x) : x \in F_q\}, \quad 1 \le i \le q-2,\ j \in Z_{q-1},$$

$$T_i(x) = \{g^j(1 + x, x, g^i + x) : j \in Z_{q-1}\} = T_i^1(x) \cup T_i^2(x), \quad x \in F_q, \text{ where}$$

$$T_i^1(x) = \left\{g^{2j}(1 + x, x, g^i + x) : 0 \le j \le \frac{q-3}{2}\right\} \quad \text{and} \quad T_i^2(x) = \left\{g^{2j+1}(1 + x, x, g^i + x) : 0 \le j \le \frac{q-3}{2}\right\}.$$

In Theorem 6.6, we will use the notices to construct an $LEDTS(q)$.

Theorem 6.6. *For* $k \in \{3, 4, 9, 11\}$, *there exists an* $LEDTS(6k+5)$.

Proof. $LEDTS(6k+5) = \{(F_{6k+5}, \mathcal{A}_x) : x \in F_{6k+5}\} \cup \{(F_{6k+5}, \mathcal{B}_i) : i \in Z_{6k+4}\} \cup \{(F_{6k+5}, \mathcal{C}_i) : i \in Z_{6k+4}\}$, where $\mathcal{A}_x = \mathcal{A}_0 + x$, $\mathcal{B}_{2t} = g^{2t}\mathcal{B}_0^0$, $\mathcal{B}_{2t+1} = g^{2t}\mathcal{B}_0^1$, $\mathcal{C}_i = g^i\mathcal{C}_0$ ($x \in F_{6k+5}, t \in Z_{3k+2}, i \in Z_{6k+4}$), and g is the primitive element of F_{6k+5}. Below, for each k, list the corresponding $\mathcal{A}_0$, $\mathcal{B}_0^0$, $\mathcal{B}_0^1$ and $\mathcal{C}_0$.

$\underline{k=3}$ $(F_{23},\ g = 5) : \mathcal{B}_0^1 = \mathcal{B}_0^0,\ \mathcal{C}_0 = (\mathcal{B}_0^0)^{-1}$,

$\mathcal{A}_0$: $T_8^1(17), T_8^2(21), T_9^1(1), T_9^2(0), T_{10}^1(17), T_{10}^2(0)$,
$T_{11}^1(13), T_{11}^2(9), T_{12}^1(3), T_{12}^2(1), T_{13}^1(20), T_{13}^2(19), T_{14}^1(5), T_{14}^2(3)$ and $(0, 0, 0), (a, 0, a), a \in F_{23}^*$.

$\mathcal{B}_0^0$: $H_1(13), H_2(19), H_3(15), H_4(12), H_5(0), H_6(1), H_7(2)$ and $(a, a, a+1), a \in F_{23}$.

$\underline{k=4}$ $(F_{29},\ g = 2) :\ \mathcal{B}_0^1 = \mathcal{B}_0^0,\ \mathcal{C}_0 = (\mathcal{B}_0^0)^{-1}$,

$\mathcal{A}_0$: $T_{10}(0), T_{11}(2), T_{12}(16), T_{13}(23), T_{14}(3), T_{15}(8), T_{16}(15), T_{17}(20), T_{18}(10),\ (0, 0, 0), (a, 0, a), a \in F_{29}^*$.

$\mathcal{B}_0^0$: $H_1(0), H_2(1), H_3(4), H_4(9), H_5(16), H_6(18), H_7(12), H_8(19), H_9(11)$ and $(a, a, a+g^{10}), a \in F_{29}$.

$\underline{k=9}$ $(F_{59},\ g = 2)$:

$\mathcal{A}_0$: $T_1^2(33), T_5^2(15), T_{15}^1(17), T_{15}^2(35), T_{17}^2(45), T_{18}^1(45), T_{20}(0), T_{21}(2), T_{22}(1), T_{23}(5), T_{24}(7), T_{25}(10)$,
$T_{26}(12), T_{27}(3), T_{28}(20), T_{29}(16), T_{30}(25), T_{31}(22), T_{33}^1(40), T_{36}^2(51), T_{41}^1(52), T_{43}^1(32)$,
$T_{48}^2(52), T_{52}^1(57), T_{53}^2(17), T_{57}^1(55)$ and $(0, 0, 0), (a, 0, a), a \in F_{59}^*$.

$\mathcal{B}_0^0$: $H_1(0), H_2(42), H_2(47), H_3(35), H_3(52), H_4(7), H_4(16), H_5(2), H_{16}(8), H_{17}(4), H_{18}(39), H_{33}(19)$,
$H_{36}(6), H_{41}(57), H_{43}(37), H_{48}(14), H_{52}(21), H_{53}(32), H_{57}(55)$ and $(a, a, a+g^{17}), a \in F_{59}$.

$\mathcal{B}_0^1$: $H_6(0), H_6(1), H_7(2), H_7(3), H_8(4), H_8(9), H_9(6), H_9(43), H_{10}(10), H_{10}(17), H_{11}(26), H_{11}(55)$,
$H_{12}(24), H_{12}(29), H_{13}(21), H_{13}(50), H_{14}(42), H_{14}(45), H_{16}(33)$ and $(a, a, a+g^{48}), a \in F_{59}$.

$\mathcal{C}_0$: $H_{19}(0), H_{32}(1), H_{34}(2), H_{35}(3), H_{37}(5), H_{38}(8), H_{39}(11), H_{40}(23), H_{42}(25), H_{44}(35), H_{45}(26), H_{46}(53)$,
$H_{47}(28), H_{49}(24), H_{50}(55), H_{51}(10), H_{54}(39), H_{55}(14), H_{56}(51)$ and $(a, a+g^{18}, a+g^{18}), a \in F_{59}$.

$\underline{k=11}$ $(F_{71},\ g = 7)$:

$\mathcal{A}_0$: $T_5^1(29), T_{14}^2(18), T_{15}^1(52), T_{19}(13), T_{24}(0), T_{25}(2), T_{26}(1), T_{27}(6), T_{28}(8), T_{29}(3), T_{30}(11), T_{31}(4)$,
$T_{32}(17), T_{33}(22), T_{34}(16), T_{35}(25), T_{36}(36), T_{37}(9), T_{38}(31), T_{41}^1(67), T_{43}^1(56), T_{44}^2(50), T_{45}^1(66)$,
$T_{46}^2(66), T_{48}^2(14), T_{50}^2(67), T_{51}^2(61), T_{61}^1(48), T_{61}^2(57), T_{68}^2(52)$ and $(0, 0, 0), (a, 0, a), a \in F_{71}^*$.

$\mathcal{B}_0^0$: $(a, a, a+g^{61}), a \in F_{71}$,
$H_1(30), H_1(51), H_2(3), H_2(38), H_3(55), H_3(68), H_4(43), H_4(64), H_5(19), H_6(42), H_6(57), H_{14}(0)$,
$H_{44}(2), H_{46}(4), H_{48}(8), H_{50}(10), H_{51}(16), H_{68}(14), H_{15}(65), H_{41}(35), H_{43}(31), H_{45}(69), H_{69}(29)$.

$\mathcal{B}_0^1$: $(a, a, a+g^{68}), a \in F_{71}$,
$H_7(2), H_7(25), H_8(4), H_8(43), H_9(8), H_9(65), H_{10}(10), H_{10}(11), H_{11}(3), H_{11}(12), H_{12}(22), H_{12}(63)$,
$H_{13}(48), H_{13}(57), H_{16}(15), H_{16}(36), H_{17}(41), H_{17}(68), H_{18}(9), H_{18}(24), H_{20}(42), H_{20}(61), H_{69}(0)$.

$\mathcal{C}_0$: $(a, a+g^2, a+g^2), a \in F_{71}$,
$H_{21}(0), H_{22}(2), H_{23}(3), H_{39}(1), H_{40}(7), H_{42}(6), H_{47}(8), H_{49}(9), H_{52}(10), H_{53}(32), H_{54}(66), H_{55}(26)$,
$H_{56}(33), H_{57}(29), H_{58}(30), H_{59}(62), H_{60}(22), H_{62}(64), H_{63}(11), H_{64}(40), H_{65}(65), H_{66}(67), H_{67}(17)$. □

Theorem 6.7. *For* $k \in \{41, 45, 47, 51, 75, 159, 161, 165, 167, 171\}$, *there exists an* $LEDTS(6k+5)$.

Proof. Let q be a prime power. Deleting two points from an $S(3, q+1, q^2+1)$ in Lemma 1.2(7), we can get a 2-$FG(3, (q, q, q+1), (q-1)^{q+1})$. Starting from such designs, let us construct some new 2-fan designs.

(1) Take $q = 7$, delete one point from a group: $FG(3, (7, 7, 8), 6^8) \to FG(3, (\{6, 7\}, \{6, 7\}, \{7, 8\}), 5^1 6^7)$;

delete three points from a group: $FG(3, (7, 7, 8), 6^8) \to FG(3, (\{6, 7\}, \{6, 7\}, \{6, 7, 8\}), 3^1 6^7)$;
delete one group and one point in another group: $FG(3, (7, 7, 8), 6^8) \to FG(3, (\{5, 6, 7\}, \{5, 6, 7\}, \{5, 6, 7, 8\}), 5^1 6^6)$.

(2) Take $q = 8$, delete an 8-block and four points from the remaining 7-group:

$$FG(3, (8, 8, 9), 7^9) \to FG(3, (\{7, 8\}, \{6, 7, 8\}, \{7, 8, 9\}), 6^8 7^1) \to FG(3, (\{6, 7, 8\}, \{5, \ldots, 8\}, \{5, \ldots, 9\}), 3^1 6^8).$$

(3) Take $q = 9$, delete 5 points from a group: $FG(3, (9, 9, 10), 8^{10}) \to FG(3, (\{8, 9\}, \{8, 9\}, \{8, 9, 10\}), 3^1 8^9)$.

(4) Take $q = 16$, delete a 16-block, delete one 15-group and three 14-groups, delete 11 points from a group:

$$\begin{aligned} FG(3, (16, 16, 17), 15^{17}) &\to FG(3, (\{15, 16\}, \{14, 15, 16\}, \{15, 16, 17\}), 14^{16} 15^1) \\ &\to FG(3, (\{11, \ldots, 16\}, \{10, \ldots, 16\}, \{7, \ldots, 17\}), 14^{13}) \\ &\to FG(3, (\{10, \ldots, 16\}, \{9, \ldots, 16\}, \{5, \ldots, 17\}), 14^{12} 3^1). \end{aligned}$$

(5) Take $q = 13$, delete i points from a group:

$$FG(3, (13, 13, 14), 12^{14}) \to FG(3, (\{12, 13\}, \{12, 13\}, \{12, 13, 14\}), 12^{13}(12-i)^1), \quad \text{for } i = 1, 3, 7, 9.$$

For $j \geq 4$, there exist $PECS^*(6^j : 0)$, $PDGDD(6^j : 5)$ and $DF(6^j)$ by Lemmas 6.1 and 3.1. Thus, from the 2-$FG(3, (\cdots), a^1 b^n)$ of type $a^1 b^n$ given in (1)–(5), we can get the corresponding $PECS((6a)^1 (6b)^n : 5)$ by Theorem 3.5. Further, there exist $LEDTS(6a+5)$ by Theorems 6.4, 6.6 and $LEDTS(6b+5, 5)$ by Theorem 6.1. We will get the desired ten $LEDTS(6k+5)$s by Theorem 3.1, where $k = a + nb$. These parameters are listed in the following table.

$a^1 b^n$	$5^1 6^7$	$3^1 6^7$	$5^1 6^6$	$3^1 6^8$	$3^1 8^9$	$3^1 14^{12}$	$3^1 12^{13}$	$5^1 12^{13}$	$9^1 12^{13}$	$11^1 12^{13}$
a	5	3	5	3	3	3	3	5	9	11
b	6	6	6	6	8	14	12	12	12	12
n	7	7	6	8	9	12	13	13	13	13
$k = a + nb$	47	45	41	51	75	171	159	161	165	167

□

Theorem 6.8. *There exists an* $LEDTS(6k+5)$ *for any integer* $k \geq 0$ *except possible* $k = 15, 23, 27, 33, 35$.

Proof. The conclusion can be obtained from Theorems 6.1–6.7 and Lemma 4.1. □

7. Conclusion

Theorem 7.1. *There exists an* $LEDTS(v)$ *for any odd* v *except possible* $v = 95, 143, 167, 203, 215$.

Proof. We can get the conclusion by Theorems 5.1, 5.2 and 6.8. □

Theorem 7.2. *There exists an* $LEDTS(v)$ *for any integer* $v \neq 4$ *except possible* $v = 95, 143, 167, 203, 215$.

Proof. We can get the conclusion by Theorem 7.1 and [9]. □

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